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6 SEM TDC MTMH (CBCS) C 13

2023

(May/June)

MATHEMATICS

(Core)

Paper : C–13

(Metric Spaces and Complex Analysis)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Real line is a metric space. State true or false. 1

(b) Write when a metric space is called complete. 1

(c) Define usual metric on R. 2

(d) Define Cauchy sequence in a metric space. 2

(e) Let X be a metric space. Show that any union of open sets in X is open. 4

Or

Show that every convergent sequence in a metric space (X, d) is a Cauchy sequence.

(f) Let X be a metric space. Show that a subset F of X is closed if and only if complement F' is open. 5

Or

In a metric space (X, d), show that each closed sphere is a closed set.

(g) Let (X, d) be a metric space and $A \subset X$. Then show that interior of A is an open set. 5

Or

Let (X, d) be a metric space and $Y \subset X$. Then show that Y is separable if X is separable.

2. (a) Define an identity function in a metric space. 1

(b) Write one example of homeomorphic spaces. 1

(c) Define uniform continuity in metric spaces. 1

(d) Define connected sets in a metric space. 2

(e) Answer any *two* questions from the following : 5×2=10

(i) Let (X, d) and (Y, r) be metric spaces and $f: X \rightarrow Y$ be a function. Then prove that f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.

(ii) Let (X, d) and (Y, r) be metric spaces and $f: X \rightarrow Y$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in X, then show that $\{f(x_n)\}$ is a Cauchy sequence in Y.

(iii) Let (X, d) be a compact metric space. Then show that a closed subset of X is compact.

3. (a) Write the condition when the complex numbers (a, b) and (c, d) are equal. 1

(b) The nth roots of unity represents the n vertices of a regular polygon. Write where the polygon is inscribed. 1

(c) Write the necessary and sufficient condition that the complex numbers represented by z_1 and z_2 become parallel. 1

(d) Find the limit of the function $f(z)$ as $z \to i$ defined by

$$f(z) = \begin{cases} z^2, & z \neq i \\ 0, & z = i \end{cases}$$ 3

Or

Write the equation $(x-3)^2 + y^2 = 9$ in terms of conjugate coordinates.

(e) Show that $\frac{d\bar{z}}{dz}$ does not exist anywhere. 4

Or

Prove that $f(z) = \begin{cases} z^2, & z \neq z_0 \\ 0, & z = z_0 \end{cases}$, where $z_0 \neq 0$ is discontinuous at $z = z_0$.

(f) Find the Cauchy-Riemann equations for an analytic function $f(z) = u + iv$, where $z = x + iy$. 5

Or

Find the equation of the circle having the line joining z_1 and z_2 as diameter.

4. *(a)* Write the point at which the function $f(z) = \frac{1+z}{1-z}$ is not analytic. 1

(b) Define singularity of a function. 2

(c) Write the statement of Cauchy's integral formula. 2

(d) Prove the equivalence of

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$ 3

(e) Find the analytic function $f(z) = u + iv$, where $u = e^x(x\cos y - y\sin y)$. 4

Or

Find the value of the integral $\int \frac{dz}{z-a}$ round a circle whose equation is $|z-a| = r$.

5. *(a)* Define radius of convergence. 1

(b) Write the necessary and sufficient condition that $\sum_{n=1}^{\infty}(a_n + ib_n)$ converges, where a_n and b_n are real. 1

(c) Define a power series. 2

(d) State and prove the fundamental theorem of algebra. 6

Or

Expand $f(z) = \log(1+z)$ in a Taylor's series about $z = 0$.

6. (a) Let R be the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n$$

Then write the radius of convergence of the series

$$\sum_{n=0}^{\infty} n a_n z^{n-1}$$ 1

(b) Choose the correct answer from the following : 1

An absolutely convergent series is

(i) divergent

(ii) convergent

(iii) oscillatory

(iv) conditionally convergent

(c) State and prove Laurent's theorem. 6

Or

Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for $1 < |z| < 3$.

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