

**5 SEM TDC MTMH (CBCS) C 12**

**2 0 2 2**

( Nov/Dec )

**MATHEMATICS**

( Core )

Paper : C-12

**( Group Theory—II )**

*Full Marks : 80*

*Pass Marks : 32*

*Time : 3 hours*

*The figures in the margin indicate full marks  
for the questions*

1. (a) Choose the correct answer for the following question : 1

An automorphism is

- (i) a homomorphism but not one-one
- (ii) a homomorphism, one-one but not onto
- (iii) one-one, onto but not homomorphism
- (iv) a homomorphism, one-one and onto

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( Turn Over )

(b) Show that a characteristic subgroup of a group  $G$  is a normal subgroup of  $G$ . Is the converse true? 2+2=4

(c) Let  $G'$  be the commutator subgroup of a group  $G$ , then prove that  $G$  is abelian if and only if  $G' = \{e\}$ . 3

(d) If  $N$  is a normal subgroup of a group  $G$ ,  $G'$  is the commutator subgroup of  $G$  and  $N \cap G' = \{e\}$ , then show that  $N \subseteq Z(G)$ . 3

(e) Show that, if  $O(\text{Aut}G) > 1$  then  $O(G) > 2$ . 3

(f) Show that the set  $I(G)$  of all inner automorphism of a group  $G$  is a subgroup of  $\text{Aut}G$ . 4

2. Answer any two of the following : 6×2=12

(a) Let  $I(G)$  be the set of all inner automorphisms on a group  $G$ , then prove that

$$I(G) \cong \frac{G}{Z(G)}$$

(b) Prove that for every positive integer  $n$ ,  $\text{Aut}(Z_n)$  is isomorphic to  $U(n)$ .

(c) Let  $R^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in R\}$ . Show that the mapping

$$\phi : (a_1, a_2, \dots, a_n) \rightarrow (-a_1, -a_2, \dots, -a_n)$$

is an automorphism of the group  $R^n$  under component wise addition.

3. (a) Find the order of the element  $(1, 1)$  in  $Z_{100} \oplus Z_{25}$ . 2

(b) Show that a group of order 4 is either cyclic or is an internal direct product of two cyclic groups of order 2 each. 3

(c) Let  $G$  and  $H$  be finite cyclic groups. Prove that  $G \oplus H$  is cyclic if and only if  $|G|$  and  $|H|$  are relatively prime. 4

(d) If  $s$  and  $t$  are relatively prime, then prove that

$$U(st) \cong U(s) \oplus U(t)$$

Or

How many elements of order 5 does  $Z_{25} \oplus Z_5$  have?

(e) If a group  $G$  is the internal direct product of a finite number of subgroups  $H_1, H_2, \dots, H_n$ , then prove that  $G$  is isomorphic to the external direct product of  $H_1, H_2, \dots, H_n$ . 6

( 4 )

Or

Let  $G$  be a finite abelian group of order  $p^n m$ , where  $p$  is a prime that does not divide  $m$  then prove that  $G = H \times K$ , where  $H = \{x \in G \mid x^{p^n} = e\}$  and  $K = \{x \in G \mid x^m = e\}$ .

4. (a) Define conjugate class of  $a$  1
- (b) If  $|G| = p^2$ , where  $p$  is a prime, then prove that  $G$  is abelian. 3
- (c) Let  $G$  be a finite group and let  $a$  be an element of  $G$ , then prove that
- $$|Cl(a)| = |G : C(a)|$$
- 3
- (d) Prove that a group of order 80 has a non-trivial normal Sylow  $p$ -subgroup. 3
- (e) Let  $G$  be a group. Prove that  $Cl(a) = \{a\}$ , if and only if  $a \in Z(G)$ . 4
- (f) Prove that no group of order 56 is simple. 5

Or

Prove that a Sylow  $p$ -subgroup of a group  $G$  is normal if and only if it is the only Sylow  $p$ -subgroup of  $G$ .

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( Continued )

( 5 )

- (g) If  $G$  is a group of order  $pq$ , where  $p$  and  $q$  are primes,  $p < q$ , and  $p$  does not divide  $q - 1$ , then prove that  $G$  is cyclic. 5

- (h) Prove that any two Sylow  $p$ -subgroups of a finite group  $G$  are conjugate in  $G$ . 6

Or

Prove that an integer of the form  $2 \cdot n$ , where  $n$  is an odd number greater than 1, is not the order of a simple group.

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