3 SEM TDC MTMH (CBCS) C 6

2022

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–6

(Group Theory—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Write each symmetry in D_3 (the set of symmetries of an equilateral triangle). 1

(b) What is the inverse of $n-1$ in $U(n)$, $n>2$? 1

(c) The set {5, 15, 25, 35} is a group under multiplication modulo 40. What is the identity element of this group? 1

(d) Let a and b belong to a group G. Find an x in G such that $xabx^{-1}=ba$. 2

(e) Show that identity element in a group is unique. 2

(f) Find the order of each element of the group $(\{0, 1, 2, 3, 4\}, +_5)$. 3

(g) Show that the four permutations I, (ab), (cd), $(ab)(cd)$ on four symbols a, b, c, d form a finite Abelian group with respect to the permutation multiplication. 5

2. *(a)* In Z_{10}, write all the elements of $<2>$. 1

(b) With the help of an example, show that union of two subgroups of a group G is not necessarily a subgroup of G. 2

(c) Define centre of an element of a group and centre of a group. 2

(d) Let G be a group and $a \in G$. Then prove that the set $H = \{a^n \mid n \in Z\}$ is a subgroup of G. 2

(e) Prove that the centre of a group G is normal subgroup of G. 4

(f) Let H and K be two subgroups of a group G. Then prove that HK is a subgroup of G if and only if $HK = KH$. 4

3. *(a)* If $|a| = 30$, find $<a^{26}>$. 1

(b) List the elements of the subgroup $<20>$ in Z_{30}. 1

(c) Find all generators of Z_6. 2

(d) Express the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix}$$

as a product of disjoint cycles. 2

(e) Find $O(f)$ where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{pmatrix}$$ 2

(f) Let a be an element of order n in a group and let k be a positive integer. Then prove that

$$<a^k> = <a^{\gcd(n,\, k)}> \text{ and } |a^k| = \frac{n}{\gcd(n,\, k)}$$ 4

Or

Prove that any two right cosets are either identical or disjoint.

(g) Prove that a group of prime order is cyclic. 3

(h) State and prove Lagrange's theorem. 5

4. *(a)* Define external direct product. 1

(b) Compute $U(8) \oplus U(10)$. Also find the product $(3, 7)(7, 9)$. 2

(c) Prove that quotient group of a cyclic group is cyclic. 3

(d) If H is a normal subgroup of a finite group G, then prove that for each $a \in G$, $O(Ha) \mid O(a)$. 4

(e) Let G be a finite Abelian group such that its order $O(G)$ is divisible by a prime p. Then prove that G has at least one element of order p. 5

Or

Let H be a subgroup of a group G such that $x^2 \in G$, $\forall x \in G$. Then prove that H is normal subgroup of G. Also prove that $\frac{G}{H}$ is Abelian.

5. *(a)* Let $(Z, +)$ and $(E, +)$ be the group of integers and even integers respectively. Show that $f: Z \to E$ defined by $f(x) = 2x$, $\forall x \in Z$ is a homomorphism. 2

(b) Prove that a homomorphic image $f: G \to G'$ is one-one if and only if $\ker f = \{e\}$, where e is the identity of G. 3

(c) Prove that every group G is isomorphic to a permutation group. 5

(d) Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. 5

Or

Let H be a normal subgroup of G and K be a subgroup of G. Then prove that

$$\frac{HK}{H} \cong \frac{K}{H \cap K}$$

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