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3 SEM TDC MTMH (CBCS) C 5

2022

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–5

(Theory of Real Functions)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State the divergence criteria of a limit of a function. 1+1=2

(b) Define cluster point of a set with an example. 1+1=2

(c) Use ε-δ definition to establish that

$$\lim_{x \to c} x^2 = c^2$$ 2

(d) Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, a cluster point of A. Show that if f has a limit, when $x \to c$, then f is bounded. 3

(e) Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, a cluster point of A. If $a \le f(x) \le b$, $\forall\ x \in A$ and $x \ne c$, and $\lim\limits_{x \to c} f(x)$ exists, then show that

$$a \le \lim_{x \to c} f(x) \le b$$ 3

(f) State and prove squeeze theorem. 1+3=4

(g) Show by using definition that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$ 3

(h) Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in A$. Then establish any *one* of the following : 3

(i) If f is continuous at $c \in A$, then given any ε-neighbourhood $V_\varepsilon(f(c))$ of $f(c)$, $\exists$ a δ-neighbourhood $V_\delta(c)$ of c, such that if $x \in A \cap V_\delta(c)$, then

$$f(x) \in V_\varepsilon(f(c)).$$

(ii) Let given any ε-neighbourhood $V_\varepsilon(f(c))$ of $f(c)$, $\exists$ a δ-neighbourhood $V_\delta(c)$ of c, such that if $x \in A \cap V_\delta(c)$, then $f(x) \in V_\varepsilon(f(c))$. Then f is continuous at $c \in A$.

(i) Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and define $|f|$ by $(|f|)(x) = |f(x)|$, $\forall\ x \in A$. Show that if f is continuous at $c \in A$, then $|f|$ is also continuous at $c \in A$. 4

Or

Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $f(x) \ge 0$, $\forall\ x \in A$. Defined $\sqrt{f}$ by $(\sqrt{f})(x) = \sqrt{f(x)}$, $\forall\ x \in A$. Show that if f is continuous at $c \in A$, then $\sqrt{f}$ is continuous at c.

(j) State and prove location roots theorem. 1+3=4

Or

Let I be a closed and bounded interval, and $f : I \to \mathbb{R}$ is continuous on I. Then show that $f : I \to \mathbb{R}$ is uniformly continuous. 4

2. (a) Define relative maximum of a real-valued function at a point. 1

(b) State the first derivative test for the relative maximum at a point of a real-valued function. 1

(c) Show that if $f : I \to \mathbb{R}$ is differentiable and $f(x) \geq 0$, $\forall\, x \in I$, then f is increasing on I. 2

(d) Using first derivative test, show that $f(x) = x^2$ has a minima at $x = 0$. 2

(e) State and prove the interior extremum theorem. 3

Or

Let $f : I \to \mathbb{R}$ be differentiable at c. If $f'(c) < 0$, then show that

$$f(x) > f(c), \quad \forall\, x \in (c - \delta, c)$$

(f) State and prove Caratheodory's theorem. 4

(g) Use mean value theorem to show that if $f(x) = \sin x$ which is differentiable, $\forall\, x \in \mathbb{R}$, then

$$|\sin x - \sin y| \leq |x - y| \quad \forall\, x, y \in \mathbb{R}$$ 4

Or

Use mean value theorem to show that

$$-x \leq \sin x \leq x \quad \forall\, x \geq 0$$

(h) State and prove the mean value theorem. 4

(i) State and prove Darboux's theorem. 4

Or

Use mean value theorem to show that

$$e^x \geq 1 + x \;\; \forall\, x \in \mathbb{R}$$

and hence show that $e^{\pi} > \pi^{e}$.

3. (a) Define a convex function on an interval and give its geometrical interpretation. 1+1=2

(b) Show that the function $f(x) = x^3$ has no relative extremum at $x = 0$. 2

(c) Show that

$$f(x) = x + \frac{1}{x};\ x > 0$$

is a convex function. 3

(d) Determine relative extrema of the function

$$f(x) = x^4 + 2x^3 - k$$

where k is a constant. 3

(e) State and prove Cauchy's mean value theorem. 5

(f) State and prove Taylor's theorem with Lagrange's form of remainder. 5

(g) Define Taylor's and Maclaurin's series. Obtain Maclaurin's series for the function $\sin x$. 2+3=5

Or

Show that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\lfloor 2n} \quad \forall\ x \in \mathbb{R}$$ 5

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