Order and degree of Partial Differential Equations (PDEs)

Partial Differential Equation (PDE) : An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a Partial Differential Equation.

Order of a PDE : The order of a PDE is defined as the order of the highest partial derivative occurring in the PDE.

Degree of a PDE : The of a PDE is the degree of the highest order derivative which occurs in it after the equation has been rationalized.

Examples :

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$ (1^{st} order & 1^{st} degree PDE)

(ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$ (3^{rd} order & 1^{st} degree PDE)

(iii) $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$ (1^{st} order & 1^{st} degree PDE)

(iv) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$ (1^{st} order & 1^{st} degree PDE)

(v) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{\frac{1}{2}}$ (2^{nd} order & 2^{nd} degree PDE)

(vi) $y\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = z\left(\frac{\partial z}{\partial y}\right)$ (1^{st} order & 2^{nd} degree PDE)

Linear and Non-linear PDEs :

A PDE is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied, otherwise it is said to be non-linear.

Examples :

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$ (Linear PDE)

(ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$ (Non-linear PDE)

(iii) $z\left(\frac{\partial z}{\partial x}\right)+\ \frac{\partial z}{\partial y}=x$ (Non-linear PDE)

(iv) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz$ (Linear PDE)

(v) $\frac{\partial^2 z}{\partial x^2}=\left(1+\frac{\partial z}{\partial y}\right)^{\frac{1}{2}}$ (Non-linear PDE)

(vi) $y\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}=z\left(\frac{\partial z}{\partial y}\right)$ (Non-linear PDE)

<u>Notations</u> **:** (i) When we consider two independent variables x and y and one dependent variable z. Then, we use the following notations in the PDEs.

$$\frac{\partial z}{\partial x}=p,\qquad \frac{\partial z}{\partial y}=q,\qquad \frac{\partial^2 z}{\partial x^2}=r,\qquad \frac{\partial^2 z}{\partial x\partial y}=s,\qquad \frac{\partial^2 z}{\partial y^2}=t$$

(ii) When we consider n independent variables $x_1, x_2, x_3, \dots \dots, x_n$ and one dependent variable z. Then, we use the following notations in the PDEs.

$$\frac{\partial z}{\partial x_1}=p_1,\quad \frac{\partial z}{\partial x_2}=p_2,\quad \frac{\partial z}{\partial x_3}=p_3,\ \dots\dots\dots\dots\dots\dots\dots,\ \frac{\partial z}{\partial x_n}=p_n$$

(iii) Partial differentiations are also denoted as :

$$\frac{\partial u}{\partial x}=u_x,\qquad \frac{\partial u}{\partial y}=u_y,\qquad \frac{\partial^2 u}{\partial x^2}=u_{xx},\qquad \frac{\partial^2 u}{\partial y^2}=u_{yy},\qquad \frac{\partial^2 u}{\partial x\partial y}=u_{xy}$$

<u>Classification of First Order PDEs</u> :

(i) Linear PDE : A first order equation $f(x,y,z,p,q)=0$ is said to be Linear PDE if it is linear in $p, q\ and\ z$, that is, if the given equation is of the form $P(x,y)p+Q(x,y)q=R(x,y)z+S(x,y)$.

Example : (i) $x^2yp+xy^2q=xyz+x^2y^2$

(ii) $p+q=z+xy$

(iii) $xp+yq=x^2y^2z+\frac{x}{y}$

(ii) Semi-linear PDE : A first order equation $f(x, y, z, p, q) = 0$ is said to be Semi-linear PDE if it is linear in $p \text{ and } q$, the co-efficients $p \text{ and } q$ are functions of $x \text{ and } y$ only, that is, if the given equation is of the form $P(x, y)p + Q(x, y)q = R(x, y, z)$.

Example : (i) $x^2yp + xy^2q = x^2y^2z^2$

(ii) $yp + xq = \frac{z^2}{xy}$

(iii) $xp + yq = \frac{x^2y^2}{z^2}$

(iii) Quasi-linear PDE : A first order equation $f(x, y, z, p, q) = 0$ is said to be Quasi-linear PDE if it is linear in $p \text{ and } q$, that is, if the given equation is of the form $P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$.

Example : (i) $x^2zp + zy^2q = xy$

(ii) $(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$

(iii) $xy^2zp + x^2yzq = (x^2y^2z^2 - 1)$

(iv) Non-linear PDE : A first order equation $f(x, y, z, p, q) = 0$ is said to be Non-linear PDE if it is not linear in $p \text{ and } q$.

Example : (i) $p^2 + q^2 = 1$

(ii) $pq = z$

(iii) $x^2p^2 + y^2q^2 = z^2$

Formation of first order partial differential equations

Rule I : Derivation of PDE by eliminating constants

Example 1. Find a PDE by eliminating a and b from

$z = ax + by + a^2 + b^2$

Solution : Given, $z = ax + by + a^2 + b^2$ ………… (i)

Differentiating (i) partially with respect to x, we get

$\frac{\partial z}{\partial x} = a$

$\Rightarrow p = a$ ……………. (ii)

Differentiating (i) partially with respect to y, we get

$\frac{\partial z}{\partial y} = b$

$\Rightarrow q = b$ ……………. (iii)

Now, putting $a = p$ and $b = q$ in (i), we get

$z = px + qy + p^2 + q^2$, which is the required PDE. **Answer**

<u>**EXERCISE 1 (A)**</u>

1. $\boldsymbol{z = A\, e^{pt} \sin px \quad (p \text{ and } A)}$

Solution : Given, $z = A\, e^{pt} \sin px$ ………… (i)

Differentiating (i) partially with respect to x, we get

$\frac{\partial z}{\partial x} = A\, e^{pt} \cos px \,.\, p$

$\Rightarrow \frac{\partial z}{\partial x} = Ape^{pt} \cos px$ ………………. (ii)

Again, differentiating (ii) partially with respect to x, we get

$\frac{\partial^2 z}{\partial x^2} = Ape^{pt}(-\sin px).\, p$

$\Rightarrow \frac{\partial^2 z}{\partial x^2} = -Ap^2e^{pt} sinpx$ ………….. (iii)

Differentiating (i) partially with respect to t, we get

$\frac{\partial z}{\partial t} = A\, pe^{pt}\, sin\, px$ …………………. (iv)

Again, differentiating (iv) partially with respect to t, we get

$\frac{\partial^2 z}{\partial t^2} = Ap^2e^{pt} sinpx$ ………………….(v)

Adding (iii) & (v), we get

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$, which is the required PDE. **Answer**

2. $\boldsymbol{z = A\, e^{-p^2 t} \cos px \quad (p\ and\ A)}$

Solution : Given, $z = A\, e^{-p^2 t} \cos px$ ………… (i)

Differentiating (i) partially with respect to x, we get

$\frac{\partial z}{\partial x} = -A\, pe^{-p^2 t} \sin px$ …………… (ii)

Again, differentiating (i) partially with respect to x, we get

$\frac{\partial^2 z}{\partial x^2} = -A\, p^2 e^{-p^2 t} \cos px$ ……………(iii)

Differentiating (i) partially with respect to t, we get

$\frac{\partial z}{\partial t} = -A\, p^2 e^{-p^2 t} \cos px$ ……………(iv)

From (iii) & (iv), we get,

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$, which is the required PDE. **Answer**

3. $\boldsymbol{z = ax^3 + by^3\ ;\ (a, b)}$

Solution : Given equation is,

$z = ax^3 + by^3$ ……………. (i)

Differentiating (i) partially with respect to x, we get

$\frac{\partial z}{\partial x} = 3ax^2$ ………………. (ii)

Differentiating (i) partially with respect to y, we get

$\frac{\partial z}{\partial y} = 3by^2$ ………………. (iii)

From (ii) and (iii), we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3ax^3 + 3by^3$$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3(ax^3 + by^3)$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \quad [From\ (i)]$

which is the required PDE. **Answer**

4. $\boldsymbol{4z = \left[ax + \frac{y}{a} + b\right]^2;\quad (a, b)}$

Solution : Given equation is,

$4z = \left[ax + \frac{y}{a} + b\right]^2$ ……………. (i)

Differentiating (i) partially with respect to x, we get

$$4\frac{\partial z}{\partial x} = 2\left(ax + \frac{y}{a} + b\right).a$$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{2}.a.\left(ax + \frac{y}{a} + b\right)$ ………….. (ii)

Differentiating (i) partially with respect to y, we get

$4\frac{\partial z}{\partial y} = 2.\left(ax + \frac{y}{a} + b\right).\frac{1}{a}$

$\Rightarrow \frac{\partial z}{\partial y} = \frac{1}{2}.\frac{1}{a}\left(ax + \frac{y}{a} + b\right)$ ……………..(iii)

Multiplying (ii) by (iii), we get

$$\frac{\partial z}{\partial x}.\frac{\partial z}{\partial y} = \frac{1}{4}.\left[ax + \frac{y}{a} + b\right]^2$$

$\Rightarrow \frac{\partial z}{\partial x}.\frac{\partial z}{\partial y} = \frac{1}{4}.4z \quad [From\ (i)]$

$\Rightarrow \frac{\partial z}{\partial x}.\frac{\partial z}{\partial y} = z$, which is the required PDE. **Answer**

5. $\boldsymbol{z = ax^2 + bxy + cy^2}$, $\quad (\boldsymbol{a, b, c})$

Solution : Given equation is,

$z = ax^2 + bxy + cy^2$ …………………….. (i)

Differentiating (i) partially with respect to x, we get

$\frac{\partial z}{\partial x} = 2ax + by$ …………………… (ii)

Differentiating (ii) partially with respect to x, we get

$\frac{\partial^2 z}{\partial x^2} = 2a$ ……………….. (iii)

Differentiating (ii) partially with respect to y, we get

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = b$

$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = b$ ………………………(iv)

Differentiating (i) partially with respect to y, we get

$\frac{\partial z}{\partial y} = bx + 2cy$ …………………… (v)

Differentiating (v) partially with respect to y, we get

$\frac{\partial^2 z}{\partial y^2} = 2c$ …………………………(vi)

From (iii), (iv) and (vi), we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2ax^2 + 2bxy + 2cy^2$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2(ax^2 + bxy + cy^2)$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2z \qquad [From\ (i)]$$

which is the required PDE. **Answer**

6. $\boldsymbol{z^2 = ax^3 + by^3 + ab,} \quad (\boldsymbol{a, b})$

Solution : Given equation is,

$z^2 = ax^3 + by^3 + ab$ …………….. (i)

Differentiating (i) partially with respect to x, we get

$2z\frac{\partial z}{\partial x} = 3ax^2$ …………………… (ii)

Differentiating (i) partially with respect to y, we get

$2z\frac{\partial z}{\partial y} = 3by^2$ …………………… (iii)

(ii) x (iii) ⇒ $4z^2 \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 9abx^2y^2$ ………………….. (iv)

From (ii) and (iii), we get

$$6x^3y^2z\frac{\partial z}{\partial x} + 6x^2y^3z\frac{\partial z}{\partial y} = 9ax^5y^2 + 9bx^2y^5 \quad \ldots\ldots\ldots\ldots\ldots \text{(v)}$$

Adding (iv) and (v), we get

$$6x^3y^2z\frac{\partial z}{\partial x} + 6x^2y^3z\frac{\partial z}{\partial y} + 4z^2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 9ax^5y^2 + 9bx^2y^5 + 9abx^2y^2$$

$$\Rightarrow 6x^3y^2z\frac{\partial z}{\partial x} + 6x^2y^3z\frac{\partial z}{\partial y} + 4z^2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 9x^2y^2(ax^3 + by^3 + ab)$$

$$\Rightarrow z\left[6x^3y^2\frac{\partial z}{\partial x} + 6x^2y^3\frac{\partial z}{\partial y} + 4z\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\right] = 9x^2y^2z^2 \qquad [From\ (i)]$$

$\Rightarrow 6x^3y^2\frac{\partial z}{\partial x} + 6x^2y^3\frac{\partial z}{\partial y} + 4z\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 9x^2y^2z$, which is required PDE.

Answer