14. Define Normal Distribution.

Solution : A random variable X is said to have a Normal Distribution with parameters μ (mean) and σ^2 (variance) if its probability density function (shortly, $p.d.f.$) is given by the following probability formula :

$$f(x;\ \mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2};\ -\infty < x < \infty,\ \ -\infty < \mu < \infty,\ \ \sigma > 0$$

15. What are the important properties of Normal Distribution ?

Solution : (i) When random variable X is normally distributed with mean μ and standard deviation σ, then X can be distributed as $N(\mu,\sigma^2)$ which is expressed by $X \sim N(\mu,\sigma^2)$.

(ii) If $X \sim N(\mu,\sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ which is a standard normal variate with $E(Z) = 0$ and $\text{Var}(Z) = 1$ and we write it as $Z \sim N(0,1)$.

(iii) The $p.d.f.$ of standard normal variate Z is given by :

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},\ -\infty < z < \infty$$

and its corresponding function $\Phi(z)$ is given by :

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \varphi(u)\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du$$

(iv) $\Phi(-z) = 1 - \Phi(z), z > 0$

(v) $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right), where\ X \sim N(\mu,\sigma^2)$

16. Prove that $\Phi(-z) = 1 - \Phi(z), z > 0$

Solution : $\Phi(-z) = P(Z \le -z) = P(Z \ge z) = 1 - P(Z \le z) = 1 - \Phi(z)$

17. Prove that $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right), where\ X \sim N(\mu,\sigma^2)$

Solution : $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right),\ \left(Z = \frac{X-\mu}{\sigma}\right)$

$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

18. Under what conditions the Normal Distribution is limiting form of binomial distribution ?

Solution : The conditions are :

(i) n, the number of trials is indefinitely large, that is, $n \to \infty$.

(ii) neither p nor q is very small.

19. What are the chief characteristics (properties) of the Normal Distribution and Normal Probability Curve ? (Important)

Solution : The normal probability curve with mean μ and standard deviation σ is given by the equation :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, -\infty < x < \infty$$

It has the following characteristics :

(a) The curve is bell-shaped and symmetrical about the line $x = \mu$.

(b) The three central tendencies i.e., Mean, Median and Mode of the distribution coincide.

(c) When x increases, $f(x)$ decreases, the maximum probability occurring at the point $x = \mu$ is given by : $[p(x)]_{max} = \frac{1}{\sigma\sqrt{2\pi}}$

(d) $\beta_1 = 0 \; and \; \beta_2 = 3$

(e) $\mu_{2r+1} = 0, (r = 0,1,2, \dots)$ and μ_{2r} = 1.3.5…(2r – 1)$\sigma^{2r}, (r = 0,1,2, \dots)$

(f) The probability $f(x)$ can never be negative and no portion of the curve lies below the x-axis.

(g) Linear combination of independent normal variates is also a normal variate.

(h) x-axis is an asymptote to the curve.

(i) The points of inflexion of the curve are : $x = \mu \pm \sigma, f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$

(j) Mean deviation about mean = $\sqrt{\frac{2}{\pi}}\sigma \simeq \frac{4}{5}\sigma\ (approx)$.