8. Derive the Moments of Binomial Distribution.

Solution : The first four moments about the origin of binomial distribution can be calculated as following ways :

$u_1{}' = \mathrm{E}(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x} = np(q+p)^{n-1}$

$= np$

$\left[\therefore \binom{n}{x} = \frac{n}{x}\binom{n-1}{x-1} = \frac{n}{x}.\frac{n-1}{x-1}.\binom{n-2}{x-2} = \frac{n}{x}.\frac{n-1}{x-1}.\frac{n-2}{x-2}\binom{n-3}{x-3}, and\ so\ on\right]$

Thus, the mean of the binomial distribution is np.

$u_2{}' = \mathrm{E}(X^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$

$= \sum_{x=0}^{n}\{x(x-1)+x\}\frac{n}{x}.\frac{n-1}{x-1}.\binom{n-2}{x-2} p^x q^{n-x}$

$= n(n-1)p^2 \left\{\sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x}\right\} + np$

$= n(n-1)p^2(q+p)^{n-2} + np$

$= n(n-1)p^2 + np$

$u_3{}' = \mathrm{E}(X^3) = \sum_{x=0}^{n} x^3 \binom{n}{x} p^x q^{n-x}$

$= \sum_{x=0}^{n}\{x(x-1)(x-2)+3x(x-1)+x\} \binom{n}{x} p^x q^{n-x}$

$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x} + 3n(n-1)p^2 \left\{\sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x}\right\}$

$+ np$

$= n(n-1)(n-2)p^3(q+p)^{n-3} + 3n(n-1)p^2(q+p)^{n-2} + np$

$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

Here, we have,

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

Let us consider,

$$x^4 = Ax(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + x$$

Putting, $x = 1,2,3$, we have the values of A, B and C.

$u_3{}' = \mathrm{E}(X^4) = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x q^{n-x}$

$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

(after simplification)

∴ Central Moments of Binomial Distribution are :

$\mu_2 = \mu_2' - {\mu_1'}^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3$

$= \{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\} - 3\{n(n-1)p^2 + np\}np + 2(np)^2$

$= np(-3np^2 + 3np + 2p^2 - 3p + 1 - 3npq)$

$= np\{3np(1-p) + 2p^2 - 3p + 1 - 3npq\}$

$= np(2p^2 - 3p + 1) = np(2p^2 - 2p + q) = npq(1-2p)$

$= npq(q + p - 2p) = npq(q-p)$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4 = npq\{1 + 3(n-2)pq\}$ (after simplification)

Hence, we have,

$$\beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq}$$

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{npq\{1+3(n-2)pq\}}{n^2p^2q^2} = \frac{1+3(n-2)pq}{npq} = 3 + \frac{1-6pq}{npq}$$

$\gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}} = \frac{1-2p}{\sqrt{npq}}$ and $\gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$

The required results.