4. Question : Derive Simpson's One-Third Rule

Answer :

Let us consider a definite integral such that

$$\mathrm{I} = \int_a^b y dx, \qquad \text{where } y = f(x)$$

Suppose, $f(x)$ is taken for $(n+1)$ equidistant values of x and those are $x_0, x_0 + h, x_0 + 2h,$ ...., $x_0 + nh$. Suppose, the range (a, b) is divided into n equal parts and width of each part is h (say), then $b - a = nh$.

Now, we assume,

$$x_0 = a$$
$$x_1 = x_0 + h = a + h$$
$$x_2 = x_1 + h = a + h + h = a + 2h$$
$$x_3 = x_2 + h = a + 2h + h = a + 3h$$

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$$x_n = a + nh = b$$

Then, we assume that $(n+1)$ ordinates $y_0, y_1, y_2,$.........,y_n which are the corresponding values of $x_0, x_1, x_2,$.........,x_n respectively, are equally spaced (equidistant).

$$\therefore \quad \mathrm{I} = \int_a^b y dx$$
$$= \int_{x_0}^{x_0+nh} y_x dx, \qquad \text{where } x_0 = a, x_n = a + nh = x_0 + nh$$

Now, we put, $u = \frac{x - x_0}{h}$

⇨ $x - x_0 = hu$

⇨ $dx = h\, du$

As, $x \to x_0$ then $u \to 0$, and $x \to x_0 + nh$ then $u \to n$

$$\therefore \quad \mathrm{I} = \int_0^n y_{x_0+hu}\ h\, du$$
$$= h \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \ldots\right] du$$
$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!}\right.$$
$$\left. + \text{.........up to } (n+1) \text{ terms}\right] \qquad \text{..................(A)}$$

Now, putting $n = 2$ in (A) and neglecting the third and higher differences, we get

$$\int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2\,!}\right]$$
$$= h\left[2y_0 + 2(y_1 - y_0) + \left(\frac{8}{3} - 2\right)\frac{1}{2}(y_2 - 2y_1 + y_0)\right]$$
$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$
$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\int_{x_0+4h}^{x_0+6h} y\,dx = \frac{h}{3}(y_4 + 4y_5 + y_6)$$

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$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n), \quad \text{when n is an even integer}$$

When n is an even integer, adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+2h} y\,dx + \int_{x_0+2h}^{x_0+4h} y\,dx + \int_{x_0+4h}^{x_0+6h} y\,dx + \ldots\ldots\ldots$$
$$\ldots + \int_{x_0+(n-2)h}^{x_0+nh} y\,dx \quad [\text{ By property of Definite Integral }]$$
$$= \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4)$$
$$+ \frac{h}{3}(y_4 + 4y_5 + y_6) + \ldots\ldots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$
$$\therefore \quad I = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_2 + y_5 + \cdots + y_{n-1})$$
$$+ 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})]$$

This formula is known as Simpson's One-Third Rule.

5. **Question : Derive Simpson's Three-Eighth Rule.**

Answer :

Let us consider a definite integral such that

$$\mathrm{I} = \int_a^b y dx, \qquad \text{where } y = f(x)$$

Suppose, $f(x)$ is taken for $(n+1)$ equidistant values of x and those are $x_0, x_0 + h, x_0 + 2h,$ …….., $x_0 + nh$. Suppose, the range (a, b) is divided into n equal parts and width of each part is h (say), then $b - a = nh$.

Now, we assume,

$$x_0 = a$$
$$x_1 = x_0 + h = a + h$$
$$x_2 = x_1 + h = a + h + h = a + 2h$$
$$x_3 = x_2 + h = a + 2h + h = a + 3h$$

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$$x_n = a + nh = b$$

Then, we assume that $(n+1)$ ordinates y_0, y_1, y_2,………,y_n which are the corresponding values of x_0, x_1, x_2,………,x_n respectively, are equally spaced (equidistant).

$$\therefore \quad \mathrm{I} = \int_a^b y dx$$
$$= \int_{x_0}^{x_0+nh} y_x dx, \qquad \text{where } x_0 = a, x_n = a + nh = x_0 + nh$$

Now, we put, $u = \frac{x - x_0}{h}$

$$\Rightarrow \quad x - x_0 = hu$$
$$\Rightarrow \quad dx = h\, du$$

As, $x \to x_0$ then $u \to 0$, and $x \to x_0 + nh$ then $u \to n$

$$\therefore \quad \mathrm{I} = \int_0^n y_{x_0+hu}\ h\, du$$
$$= h \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \ldots \right] du$$
$$= h \left[n y_0 + \frac{n^2}{2}\,\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} \right.$$
$$\left. + \ldots\ldots\ \text{up to } (n+1) \text{ terms} \right] \qquad \ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Now, putting $n = 3$ in (A) and neglecting the fourth and higher differences, we get

$$\int_{x_0}^{x_0+3h} y dx = h\left[3y_0 + \frac{3^2}{2}\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2\,!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3\,!}\right]$$

$$= h\,[3y_0 + \frac{9}{2}(y_1 - y_0) + \left(9 - \frac{9}{2}\right)\frac{1}{2}(y_2 - 2y_1 + y_0)$$
$$+ \left(\frac{81}{4} - 27 + 9\right)\frac{1}{6}(y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= h\,[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0)$$
$$+ \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} y dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

$$\int_{x_0+6h}^{x_0+9h} y dx = \frac{3h}{8}(y_6 + 3y_7 + 3y_8 + y_9)$$

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$$\int_{x_0+(n-3)h}^{x_0+nh} y dx = \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

when n is a multiple of 3.

Adding all these integrals, when n is a multiple of 3, we get

$$\int_{x_0}^{x_0+nh} y dx = \int_{x_0}^{x_0+3h} y dx + \int_{x_0+3h}^{x_0+6h} y dx + \int_{x_0+6h}^{x_0+9h} y dx + \ldots\ldots\ldots$$
$$\ldots + \int_{x_0+(n-3)h}^{x_0+nh} y dx \quad \text{[By property of Definite Integral]}$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3) + \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$
$$+ \frac{3h}{8}(y_6 + 3y_7 + 3y_8 + y_9) + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots + \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

$$\therefore \quad I = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots\ldots + y_{n-2} + y_{n-1})$$
$$+ 2(y_3 + y_6 + y_9 + \ldots\ldots\ldots + y_{n-3})]$$

This formula is known as Simpson's Three-Eighth Rule.