Numerical Quadrature

1. **Question : Define Numerical Quadrature.**

   Answer :

   The process which is applied to find out the value of a definite integral from a set of numerical values of the integrand i.e., the function of a single variable, is known as Numerical Quadrature.

2. **Question : Derive General Quadrature Formula.**

   Answer :

   Let us consider a definite integral such that

   $$\mathrm{I} = \int_a^b y\,dx, \qquad \text{where } y = f(x)$$

   Suppose, $f(x)$ is taken for $(n+1)$ equidistant values of x and those are $x_0, x_0 + h, x_0 + 2h, \ldots\ldots, x_0 + nh$. Suppose, the range (a, b) is divided into n equal parts and width of each part is h (say), then $b - a = nh$.

   Now, we assume,

   $$\begin{aligned} x_0 &= a \\ x_1 &= x_0 + h = a + h \\ x_2 &= x_1 + h = a + h + h = a + 2h \\ x_3 &= x_2 + h = a + 2h + h = a + 3h \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= a + nh = b \end{aligned}$$

   Then, we assume that $(n+1)$ ordinates $y_0, y_1, y_2, \ldots\ldots\ldots, y_n$ which are the corresponding values of $x_0, x_1, x_2, \ldots\ldots\ldots, x_n$ respectively, are equally spaced (equidistant).

   $$\therefore \qquad \mathrm{I} = \int_a^b y\,dx$$

$$= \int_{x_0}^{x_0+nh} y_x dx, \quad \text{where } x_0 = a, x_n = a + nh = x_0 + nh$$

Now, we put, $u = \frac{x - x_0}{h}$

⇨ $x - x_0 = hu$

⇨ $dx = h\, du$

As, $x \to x_0$ then $u \to 0$, and $x \to x_0 + nh$ then $u \to n$

$\therefore$ $\mathrm{I} = \int_0^n y_{x_0+hu}\, h\, du$

$$= h \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2\,!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!} \Delta^3 y_0 + \ldots \right] du$$

$$= h \left[n y_0 + \frac{n^2}{2} \Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2} \right) \frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2 \right) \frac{\Delta^3 y_0}{3!} \right.$$

$\left. + \ldots\ldots\text{up to } (n+1) \text{ terms} \right]$ ....(A)

The formula (A) is known as General Quadrature Formula.

3. Question : Derive Trapezoidal Rule.

Answer :

Let us consider a definite integral such that

$$\mathrm{I} = \int_a^b y dx, \qquad \text{where } y = f(x)$$

Suppose, $f(x)$ is taken for $(n + 1)$ equidistant values of x and those are $x_0, x_0 + h, x_0 + 2h, \ldots\ldots, x_0 + nh$. Suppose, the range (a, b) is divided into n equal parts and width of each part is h (say), then $b - a = nh$.

Now, we assume,

$x_0 = a$

$x_1 = x_0 + h = a + h$

$x_2 = x_1 + h = a + h + h = a + 2h$

$x_3 = x_2 + h = a + 2h + h = a + 3h$

....

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$x_n = a + nh = b$

Then, we assume that $(n+1)$ ordinates $y_0, y_1, y_2,\ldots\ldots\ldots,y_n$ which are the corresponding values of $x_0, x_1, x_2,\ldots\ldots\ldots,x_n$ respectively, are equally spaced (equidistant).

$$\therefore \quad \mathrm{I} = \int_a^b y dx$$
$$= \int_{x_0}^{x_0+nh} y_x dx, \qquad \text{where } x_0 = a, x_n = a + nh = x_0 + nh$$

Now, we put, $u = \frac{x-x_0}{h}$

$$\Rightarrow \quad x - x_0 = hu$$
$$\Rightarrow \quad dx = h\, du$$

As, $x \to x_0$ then $u \to 0$, and $x \to x_0 + nh$ then $u \to n$

$$\therefore \quad \mathrm{I} = \int_0^n y_{x_0+hu}\ h\, du$$
$$= h \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \ldots\right] du$$
$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!}\right.$$
$$\left. + \ldots\ldots\ldots \text{up to } (n+1) \text{ terms}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Now, putting $n = 1$ in (A) and neglecting the second and higher differences, we get

$$\int_{x_0}^{x_0+h} y dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right], \qquad \text{where } I = \int_{x_0}^{x_0+nh} y dx$$
$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$
$$= h\left(\frac{y_0 + y_1}{2}\right)$$

Secondly,

$$\int_{x_0+h}^{x_0+2h} y dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right]$$
$$= h\left[y_1 + \frac{1}{2}(y_2 - y_1)\right]$$
$$= h\left(\frac{y_1 + y_2}{2}\right)$$

Similarly,

$$\int_{x_0+2h}^{x_0+3h} y dx = h\left(\frac{y_2 + y_3}{2}\right)$$

$$\int_{x_0+3h}^{x_0+4h} y dx = h\left(\frac{y_3 + y_4}{2}\right)$$

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$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left(\frac{y_{n-1}+y_n}{2}\right)$$

Adding these n integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+h} y\,dx + \int_{x_0+h}^{x_0+2h} y\,dx + \int_{x_0+2h}^{x_0+3h} y\,dx + \ldots\ldots\ldots$$

$$\ldots + \int_{x_0+(n-1)h}^{x_0+nh} y\,dx \quad \text{[By property of Definite Integral]}$$

$$= h\left(\frac{y_0+y_1}{2}\right) + h\left(\frac{y_1+y_2}{2}\right) + h\left(\frac{y_2+y_3}{2}\right) + \ldots + h\left(\frac{y_{n-1}+y_n}{2}\right)$$

$$\therefore \quad I = h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + \ldots\ldots\ldots + y_{n-1})\right]$$

This formula is known as Trapezoidal Rule.