Lagrange's Interpolation Formula with Unequal Intervals :

Suppose, $y = f(x)$ is a given function.

Let us consider, $f(x_0), f(x_1), f(x_2), \ldots\ldots\ldots\ldots\ldots, f(x_n)$ are the values of the function $y = f(x)$ corresponding to the arguments $x_0, x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n$ respectively, not necessarily equally spaced.

We assume, $P_n(x)$ is a polynomial in x of degree n such that

$$\begin{aligned}
P_n(x) = {} & A_0(x - x_1)(x - x_2)(x - x_3) \ldots\ldots\ldots(x - x_n) \\
& + A_1(x - x_0)(x - x_2)(x - x_3) \ldots\ldots\ldots(x - x_n) \\
& + A_2(x - x_0)(x - x_1)(x - x_3) \ldots\ldots\ldots(x - x_n) \\
& + A_3(x - x_0)(x - x_1)(x - x_2) \ldots\ldots\ldots(x - x_n) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + A_n(x - x_0)(x - x_1)(x - x_2) \ldots\ldots\ldots(x - x_{n-1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)
\end{aligned}$$

Where $A_0, A_1, A_3, \ldots \ldots\ldots\ldots, A_n$ are $(n + 1)$ constants to be determined.

To determine the constants $A_0, A_1, A_3, \ldots \ldots\ldots\ldots, A_n$, we assume

$$P_n(x_0) = f(x_0)$$

$$P_n(x_1) = f(x_1)$$

$$P_n(x_2) = f(x_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$P_n(x_n) = f(x_n)$$

Now, putting successively $x = x_0, x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_n$ in (1), we get

$$f(x_0) = A_0(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \ldots\ldots\ldots(x_0 - x_n)$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \ldots\ldots\ldots(x_0 - x_n)}$$

Similarly, we have

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\ \dots\dots\dots.(x_1 - x_n)}$$

$$A_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\ \dots\dots\dots.(x_2 - x_n)}$$

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$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\ \dots\dots\dots.(x_n - x_{n-1})}$$

Now, putting these values of the constants in (1), we get

$$P_n(x) = \frac{(x - x_1)(x - x_2)(x - x_3)\ \dots\dots\dots.(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)\ \dots\dots\dots.(x_0 - x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\ \dots\dots\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ \dots\dots\dots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ \dots\dots\dots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ \dots\dots\dots(x_2-x_n)} f(x_2)$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\ \dots\dots\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ \dots\dots\dots(x_n-x_{n-1})} f(x_n) \quad \dots\dots\dots\dots\dots\dots\dots (2)$$

This formula (2) is known as Lagrange's Interpolation Formula with unequal intervals.

Worked out Example : The following values of the function $f(x)$ for values of x are given as

$$f(1) = 3, \qquad f(2) = 9, \quad f(4) = 15, \qquad f(7) = 20$$

Find the value of $f(5)$.

Solution : Applying Lagrange's Formula, we get

$$\frac{f(x)}{(x-1)(x-2)(x-4)(x-7)} = \frac{f(1)}{(1-2)(1-4)(1-7)(x-1)}$$

$$+\frac{f(2)}{(2-1)(2-4)(2-7)(x-2)}$$

$$+\frac{f(4)}{(4-1)(4-2)(4-7)(x-4)}$$

$$+\frac{f(7)}{(7-1)(7-2)(7-4)(x-7)}$$

$$\Rightarrow \frac{f(x)}{(x-1)(x-2)(x-4)(x-7)} = \frac{3}{(-18)(x-1)} + \frac{9}{10(x-2)} + \frac{15}{(-18)(x-4)} + \frac{20}{90(x-7)}$$

$$\Rightarrow \frac{f(x)}{(x-1)(x-2)(x-4)(x-7)} = \frac{-1}{6(x-1)} + \frac{9}{10(x-2)} + \frac{-5}{6(x-4)} + \frac{2}{9(x-7)}$$

$$\Rightarrow f(x) = \frac{(44x-34)(x-4)(x-7)}{60} + \frac{(-33x+267)(x-1)(x-2)}{54}$$

Putting $x = 5$, we get

$$f(x) = \frac{186 \text{ x } 1 \text{ x } (-2)}{60} + \frac{102 \text{ x } 4 \text{ x } 3}{54}$$

$$= -6.2 + 22.67$$

$$= 16.47$$