Solution of linear PDEs by Lagrange's Method (Type – 3 based on Rule III)

Example (1) : Solve

$$\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$$

Solution : Given PDE is,

$\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$ ……… (i)

The Lagrange's auxiliary equations for (i) are,

$$\frac{dx}{\{(b-c)/a\}yz} = \frac{dy}{\{(c-a)/b\}zx} = \frac{dz}{\{(a-b)/c\}xy}$$

$\Rightarrow \frac{adx}{(b-c)yz} = \frac{bdy}{(c-a)zx} = \frac{cdz}{(a-b)xy}$………………….. (ii)

Choosing x, y, z as multipliers, each fraction for (ii)

$$= \frac{axdx+bydy+czdz}{(b-c)xyz+(c-a)yzx+(a-b)xyz} \quad = \frac{axdx+bydy+czdz}{xyz\{(b-c)+(c-a)+(a-b)\}}$$

$$= \frac{axdx + bydy + czdz}{0}$$

$\therefore\ axdx + bydy + czdz = 0$

$\Rightarrow \int axdx + \int bydy + \int czdz = \frac{c_1}{2}$, where c_1 is an integrating constant.

$$\Rightarrow \frac{ax^2}{2} + \frac{by^2}{2} + \frac{cz^2}{2} = \frac{c_1}{2}$$

$\therefore\ ax^2 + by^2 + cz^2 = c_1$ …………………….. (iii)

Choosing ax, by, cz as multipliers, each fraction for (ii)

$$= \frac{a^2xdx+b^2ydy+c^2zdz}{a(b-c)xyz+b(c-a)yzx+c(a-b)xyz} \qquad = \frac{a^2xdx+b^2ydy+c^2zdz}{xyz\{a(b-c)+b(c-a)+c(a-b)\}}$$

$$= \frac{a^2xdx + b^2ydy + c^2zdz}{0}$$

$\therefore\ a^2xdx + b^2ydy + c^2zdz = 0$

$\Rightarrow \int a^2xdx + \int b^2ydy + \int c^2zdz = \frac{c_1}{2}$, where c_2 is an integrating constant.

$\Rightarrow \frac{a^2x^2}{2} + \frac{b^2y^2}{2} + \frac{c^2z^2}{2} = \frac{c_2}{2}$

$\therefore\ a^2x^2 + b^2y^2 + c^2z^2 = c_2$ …………………….. (iv)

From (iii) and (iv), the general solution is

$\varphi(c_1, c_2) = 0$

$i.e., \varphi(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2) = 0$, where φ is an arbitrary function.

Exercise 2(C)

1. $\boldsymbol{x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)}$

Solution : Given PDE is

$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ …………… (i)

The Lagrange's auxiliary equations for (i) are,

$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$ ……………… (ii)

Choosing x, y, z as multipliers, each fraction for (ii)

$= \frac{xdx+ydy+zdz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)} = \frac{xdx+ydy+zdz}{0}$

$\therefore\ xdx + ydy + zdz = 0$

$\Rightarrow \int xdx + \int ydy + \int zdz = \frac{c_1}{2}$, where c_1 is an integrating constant.

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$\therefore\ x^2 + y^2 + z^2 = c_1$ …………………….. (iii)

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction for (ii)

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{(y^2-z^2)+(z^2-x^2)+(x^2-y^2)} \qquad = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = logc_2$, where c_2 is an integrating constant.

$\Rightarrow \log x + \log y + \log z = \log c_2$

$\Rightarrow log(xyz) = \log c_2$

$\therefore xyz = c_2$ …………………….. (iv)

From (iii) and (iv), the general solution is

$\varphi(c_1, c_2) = 0$

$i.e., \varphi(x^2 + y^2 + z^2, xyz) = 0$, where φ is an arbitrary function. **Answer**

2. $z(xp - yq) = y^2 - x^2$

Solution : Given PDE is

$z(xp - yq) = y^2 - x^2$

$\Rightarrow zxp - yzq = y^2 - x^2$ ………….. (i)

The Lagrange's auxiliary equations for (i) are,

$\frac{dx}{zx} = \frac{dy}{-yz} = \frac{dz}{y^2-x^2}$ ……………… (ii)

Choosing x, y, z as multipliers, each fraction for (ii)

$$= \frac{xdx+ydy+zdz}{zx^2-y^2z+y^2z-zx^2} = \frac{xdx+ydy+zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

$\Rightarrow \int xdx + \int ydy + \int zdz = \frac{c_1}{2}$, where c_1 is an integrating constant.

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$\therefore\ x^2 + y^2 + z^2 = c_1$ …………………….. (iii)

Choosing $\frac{1}{x}, \frac{1}{y}, 0$ as multipliers, each fraction for (ii)

$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + 0dz}{z - z + 0} \qquad = \frac{\frac{1}{x}dx + \frac{1}{y}dy}{0}$

$$\therefore\ \frac{1}{x}dx + \frac{1}{y}dy = 0$$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy = \log c_2$, where c_2 is an integrating constant.

$\Rightarrow \log x + \log y = \log c_2$

$\Rightarrow\ log(xy) = \log c_2$

$\therefore xy = c_2$…………………….. (iv)

From (iii) and (iv), the general solution is

$\varphi(c_1, c_2) = 0$

$i.e., \varphi(x^2 + y^2 + z^2, xy) = 0$, where φ is an arbitrary function. **Answer**

3. $(y^2 + z^2)p - xyq + xz = 0$

Solution : Given PDE is

$(y^2 + z^2)p - xyq + xz = 0$

$\Rightarrow (y^2 + z^2)p - xyq = -xz$ ……………. (i)

The Lagrange's auxiliary equations for (i) are,

$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$ ……………… (ii)

Choosing x, y, z as multipliers, each fraction for (ii)

$$= \frac{xdx+ydy+zdz}{xy^2+xz^2-xy^2-xz^2} = \frac{xdx+ydy+zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$

$\Rightarrow \int xdx + \int ydy + \int zdz = \frac{c_1}{2}$, where c_1 is an integrating constant.

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$\therefore\ x^2 + y^2 + z^2 = c_1$ ……………………. (iii)

Choosing $0, \frac{1}{y}, \frac{-1}{z}$ as multipliers, each fraction for (ii)

$$= \frac{0dx+\frac{1}{y}dy-\frac{1}{z}dz}{0+x-x} \qquad = \frac{\frac{1}{y}dy-\frac{1}{z}dz}{0}$$

$\therefore\ \frac{1}{y}dy - \frac{1}{z}dz = 0$

$\Rightarrow \int \frac{1}{y}dy - \int \frac{1}{z}dz = \log c_2$, where c_2 is an integrating constant.

$\Rightarrow \log y - \log z = \log c_2$

$\Rightarrow\ log\left(\frac{y}{z}\right) = \log c_2$

$\therefore \frac{y}{z} = c_2$…………………….. (iv)

From (iii) and (iv), the general solution is

$\varphi(c_1, c_2) = 0$

$i.e., \varphi\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0$, where φ is an arbitrary function. **Answer**

4. $yp - xq = 2x - 3y$

Solution : Given PDE is

$yp - xq = 2x - 3y$ ……………. (i)

The Lagrange's auxiliary equations for (i) are,

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x-3y} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Choosing $x, y, 0$ as multipliers, each fraction for (ii)

$$= \frac{xdx+ydy+0dz}{xy-xy} \qquad = \frac{xdx+ydy}{0}$$

$$\therefore\ xdx + ydy = 0$$

$\Rightarrow \int xdx + \int ydy = \frac{c_1}{2}$, where c_1 is an integrating constant.

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$\therefore\ x^2 + y^2 = c_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Choosing 3,2,1 as multipliers, each fraction for (ii)

$$= \frac{3dx+2dy+dz}{3y-2x+2x-3y} \qquad = \frac{3dx+2dy+dz}{0}$$

$$\therefore\ 3dx + 2dy + dz = 0$$

$\Rightarrow \int 3dx + \int 2dy + \int dz = \mathrm{c}_2$, where c_2 is an integrating constant.

$$\therefore 3\mathrm{x} + 2\mathrm{y} + \mathrm{z} = \mathrm{c}_2 \quad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

From (iii) and (iv), the general solution is

$$\varphi(c_1, c_2) = 0$$

$i.e., \varphi(x^2 + y^2, 3\mathrm{x} + 2\mathrm{y} + \mathrm{z}) = 0$, where φ is an arbitrary function. **Answer**

5. $\boldsymbol{x^2(y-z)p + y^2(z-x)q = z^2(x-y)}$

Solution : Given PDE is

$$x^2(y-z)p + y^2(z-x)q = z^2(x-y) \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The Lagrange's auxiliary equations for (i) are,

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction for (ii)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)} \quad = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \ \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = logc_1,$ where c_1 is an integrating constant.

$$\Rightarrow logx + logy + logz = logc_1$$

$$\Rightarrow \ \log(xyz) = logc_1$$

$$\therefore \ xyz = c_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Choosing $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, each fraction for (ii)

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)} \quad = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \ \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$\Rightarrow \int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = -c_2,$ where c_2 is an integrating constant.

$$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -c_2$$

$$\therefore \ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -c_2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

From (iii) and (iv), the general solution is

$$\varphi(c_1, c_2) = 0$$

$i.e., \varphi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$, where φ is an arbitrary function. **Answer**