Solution by Lagrange's Method (Type 2 based on Rule II)

EXERCISE 2(B)

1. $\boldsymbol{p - 2q = 3x^2 sin(y + 2x)}$

Solution : Given PDE is,

$p - 2q = 3x^2 sin(y + 2x)$ …………………. (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 sin(y+2x)}$ ……………………. (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{1} = \frac{dy}{-2}$$

$\Rightarrow -2dx = dy$

$\Rightarrow \int -2dx + c_1 = \int dy$

$\Rightarrow -2x + c_1 = y$

$\Rightarrow c_1 = y + 2x$ ……………………… (iii)

From 1st and 3rd fractions of (ii), we get

$$\frac{dx}{1} = \frac{dz}{3x^2 sin(y + 2x)}$$

$\Rightarrow dx = \frac{dz}{3x^2 sinc_1}$ $\quad [From\ (iii), c_1 = y + 2x]$

$\Rightarrow 3x^2 sinc_1 dx = dz$

$\Rightarrow sinc_1 \int 3x^2 dx = \int dz + c_2,$ where c_2 is an integrating constant.

$\Rightarrow x^3 sin(y + 2x) = z + c_2$

$\Rightarrow x^3 sin(y + 2x) - z = c_2$

$\therefore$ *The required general solution will be*

$c_2 = \varphi(c_1)$

$x^3 sin(y + 2x) - z = \varphi(y + 2x), \ \varphi$ *is an arbitrary function.* **Answer**

2. $\boldsymbol{p - q = \frac{z}{x+y}}$

Solution : Given PDE is,

$p - q = \frac{z}{x+y}$ …………………… (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\frac{z}{x+y}}$ …………………….. (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{1} = \frac{dy}{-1}$$

$\Rightarrow -dx = dy$

$\Rightarrow \int -dx + c_1 = \int dy$

$\Rightarrow -x + c_1 = y$

$\Rightarrow c_1 = x + y$ ……………………… (iii)

From 1st and 3rd fractions of (ii), we get

$$\frac{dx}{1} = \frac{dz}{\frac{z}{x+y}}$$

$\Rightarrow dx = \frac{c_1 dz}{z}$ $\quad [From\ (iii), c_1 = x + y]$

$\Rightarrow \int dx = \int \frac{c_1 dz}{z} + c_2,$ where c_2 is an integrating constant

$\Rightarrow c_2 = \; x - c_1 logz$

$\Rightarrow c_2 = \; x - (x + y)logz$

$\therefore The\ required\ general\ solution\ will\ be$

$c_2 = \; \varphi(c_1)$

$x - (x + y)logz = \; \varphi(x + y), \; \varphi\ is\ an\ arbitrary\ function.$ **Answer**

3. $\boldsymbol{xy^2p - y^3q + axz = 0}$

Solution : Given PDE is,

$xy^2p - y^3q + axz = 0$

$\Rightarrow xy^2p - y^3q = -axz$ …………………. (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{xy^2} = \frac{dy}{-y^3} = \frac{dz}{-axz}$ ……………………. (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{xy^2} = \frac{dy}{-y^3}$$

$\Rightarrow \frac{dx}{x} = \frac{dy}{-y}$

$\Rightarrow \int \frac{dx}{x} = -\int \frac{dy}{y} + logc_1$, where $c_1$is an integrating constant.

$\Rightarrow logx = -logy + logc_1$

$\Rightarrow logc_1 = logx + logy = log(xy)$

$\Rightarrow c_1 = xy$ ……………………... (iii)

From 2nd and 3rd fractions of (ii), we get

$$\frac{dy}{-y^3} = \frac{dz}{-axz}$$

$\Rightarrow \frac{dy}{y^3} = \frac{dz}{a.\frac{c_1}{y}.z}$ [From (iii), $x = \frac{c_1}{y}$]

$\Rightarrow \frac{dy}{y^4} = \frac{dz}{ac_1 z}$

$\Rightarrow ac_1.\frac{dy}{y^4} = \frac{dz}{z}$

$\Rightarrow ac_1 \int \frac{dy}{y^4} + c_2 = \int \frac{dz}{z}$

$\Rightarrow a\,xy.\left(-\frac{1}{3y^3}\right) + c_2 = logz$ [From (iii), $c_1 = xy$]

$\Rightarrow -\frac{ax}{3y^2} + c_2 = logz$

$\Rightarrow c_2 = logz + \frac{ax}{3y^2}$ ………………… (iv)

From, (iii) and (iv), we get the general solution of the given equation is,

$c_2 = \varphi(c_1)$

i.e., $logz + \frac{ax}{3y^2} = \varphi(xy)$ **Answer**

5. (a) $z(p - q) = z^2 + (x + y)^2$

Solution : Given PDE is,

$z(p - q) = z^2 + (x + y)^2$

$\Rightarrow zp - zq = z^2 + (x + y)^2$……………. (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$ …………………….. (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{z} = \frac{dy}{-z}$$

$\Rightarrow dx + dy = 0$

$\Rightarrow \int dx + \int dy = c_1$ where c_1 is an integrating constant.

$\Rightarrow c_1 = x + y$ …………………….. (iii)

From 2nd and 3rd fractions of (ii), we get

$$\frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

$\Rightarrow \frac{dy}{-z} = \frac{dz}{z^2+{c_1}^2}$ [From (iii), $c_1 = x + y$]

$\Rightarrow -dy = \frac{zdz}{z^2+{c_1}^2}$

$\Rightarrow -\int dy + \frac{1}{2} log c_2 = \frac{1}{2}. \int \frac{2zdz}{z^2+{c_1}^2}$ where c_2 is an integrating constant.

$\Rightarrow -y + \frac{1}{2} log c_2 = \frac{1}{2} log(z^2 + {c_1}^2)$

$\Rightarrow -2y + log c_2 = log\{(z^2 + {c_1}^2)\}$

$\Rightarrow log c_2 = 2y + log\{(z^2 + {c_1}^2)\}$

$\Rightarrow c_2 = e^{2y}[z^2 + {c_1}^2]$

$\Rightarrow c_2 = e^{2y}[z^2 + (x+y)^2]$ ……………………………………….(iv)

From (iii) and (iv), the general solution is

$e^{2y}[z^2 + (x+y)^2] = \varphi(x+y)$, where φ is an arbitrary function. **Answer**

(b) $z(p+q) = z^2 + (x-y)^2$

Solution : Given PDE is,

$z(p+q) = z^2 + (x-y)^2$

$\Rightarrow zp + zq = z^2 + (x-y)^2$................ (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{z} = \frac{dy}{z} = \frac{dz}{z^2+(x-y)^2}$ .... (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{z} = \frac{dy}{z}$$

$\Rightarrow dx - dy = 0$

$\Rightarrow \int dx - \int dy = c_1$ where $c_1$is an integrating constant.

$\Rightarrow c_1 = x - y$ .... (iii)

From 2nd and 3rd fractions of (ii), we get

$$\frac{dy}{z} = \frac{dz}{z^2 + (x-y)^2}$$

$\Rightarrow \frac{dy}{z} = \frac{dz}{z^2+{c_1}^2}$ [From (iii), $c_1 = x + y$]

$\Rightarrow dy = \frac{zdz}{z^2+{c_1}^2}$

$\Rightarrow \int dy + \frac{1}{2} log c_2 = \frac{1}{2}. \int \frac{2zdz}{z^2+{c_1}^2}$ where c_2 is an integrating constant.

$\Rightarrow y + \frac{1}{2} log c_2 = \frac{1}{2} log(z^2 + {c_1}^2)$

$\Rightarrow 2y + log c_2 = log\{(z^2 + {c_1}^2)\}$

$\Rightarrow log c_2 = -2y + log\{(z^2 + {c_1}^2)\}$

$\Rightarrow c_2 = e^{-2y}[z^2 + {c_1}^2]$

$\Rightarrow c_2 = e^{-2y}[z^2 + (x-y)^2]$ ....(iv)

From (iii) and (iv), the general solution is

$e^{-2y}[z^2 + (x-y)^2]=\varphi(x-y)$, where φ is an arbitrary function. **Answer**

8. $\boldsymbol{zp - zq = x + y}$

Solution : Given PDE is

$zp - zq = x + y$ ………………………… (i)

The Lagrange's auxiliary equations for (i) will be

$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{x+y}$ …………………….. (ii)

From 1st & 2nd fractions of (ii), we get

$$\frac{dx}{z} = \frac{dy}{-z}$$

$\Rightarrow dx = -dy$

$\Rightarrow \int dx = -\int dy + c_1$, where c_1 is an integrating constant.

$\Rightarrow x = -y + c_1$

$\Rightarrow c_1 = x + y$ …………………………….. (iii)

From 1st & 3rd fractions of (ii), we get

$$\frac{dx}{z} = \frac{dz}{x+y}$$

$\Rightarrow \frac{dx}{z} = \frac{dz}{c_1}$ $[from\ (i)]$

$\Rightarrow c_1 dx = zdz$

$\Rightarrow c_1 \int dx = \int zdz + \frac{c_2}{2}$, where c_2 is an integrating constant

$\Rightarrow c_1 x = \frac{z^2}{2} + \frac{c_2}{2}$

$\Rightarrow c_2 = 2c_1 x - z^2$

$\Rightarrow c_2 = 2(x+y)x - z^2$ ………………….. (iv)

From, (iii) and (iv), we get the general solution of the given equation is,

$c_2 = \varphi(c_1)$

i.e., $2(x+y)x - z^2 = \varphi(x+y)$, where φ is an arbitrary function. **Answer**

9. $\boldsymbol{xyp + y^2q + 2x^2 - xyz = 0}$

Solution : Given PDE is

$xyp + y^2q + 2x^2 - xyz = 0$ .... (i)

The Lagrange's auxiliary equations for (i) will be

$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{2x^2 - xyz}$ .... (ii)

From 1st & 2nd fractions of (ii), we get

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$

$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y} + log c_1$, where c_1 is an integrating constant.

$\Rightarrow \log \frac{x}{y} = log c_1$

$\Rightarrow c_1 = \frac{x}{y}$ .... (iii)

From 2nd & 3rd fractions of (ii), we get

$$\frac{dy}{y^2} = \frac{dz}{2x^2 - xyz}$$

$\Rightarrow \frac{dy}{y^2} = \frac{dz}{2{c_1}^2 y^2 - c_1 y.yz}$ [from (iii), $x = c_1 y$]

$\Rightarrow dy = \frac{dz}{c_1(2c_1 - z)}$

$\Rightarrow c_1 dy = \frac{dz}{2c_1 - z}$

Let us take, $2c_1 - z = u$

$\Rightarrow dz = -du$

$\Rightarrow \int \frac{dz}{2c_1 - z} = -\int \frac{du}{u} = logu$

$= -log(2c_1 - z)$

$\Rightarrow c_1 \int dy = \int \frac{dz}{2c_1 - z} + c_2$, where c_2 is an IC

$\Rightarrow c_1 y = -log(2c_1 - z) + c_2$

$\Rightarrow c_1 y + log(2c_1 - z) = c_2$

$\Rightarrow \frac{x}{y}.y + log\left(\frac{2x}{y} - z\right) = c_2$

$\Rightarrow x + \ log\left(\frac{2x}{y} - z\right) = c_2$ ………………………… (iv)

From, (iii) and (iv), we get the general solution of the given equation is,

$c_2 = \varphi(c_1)$

i.e., $x + \ log\left(\frac{2x}{y} - z\right) = \ \varphi\left(\frac{x}{y}\right)$, where φ is an arbitrary function. **Answer**