Lagrange's method solving Linear Partial Differential Equations

Type 1 based on Rule I

Recommended Book : Advanced Differential Equations (M.D.Raisinghania)

Exercise 2 (A)

Solve the following PDEs :

1. $(-\boldsymbol{a}+\boldsymbol{x})\boldsymbol{p}+(-\boldsymbol{b}+\boldsymbol{y})\boldsymbol{q}=(-\boldsymbol{c}+\boldsymbol{z})$

$Solution$: Given PDE is

$(-a+x)p+(-b+y)q=(-c+z)$ .... (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{x-a}=\frac{dy}{y-b}=\frac{dz}{z-c}$ .... (ii)

From 1st and 2nd fractions, we get

$$\frac{dx}{x-a}=\frac{dy}{y-b}$$

$\Rightarrow \int\frac{dx}{x-a}=\int\frac{dy}{y-b}+logc_1$ where $logc_1$ is an integrating constant

$\Rightarrow \log(x-a)=\log(y-b)+logc_1$

$\Rightarrow \log(x-a)-\log(y-b)= logc_1$

$\Rightarrow logc_1=log\frac{x-a}{y-b}$

$$\therefore c_1=\frac{x-a}{y-b}$$

From 1st and 3rd fractions, we get

$$\frac{dx}{x-a}=\frac{dz}{z-c}$$

$\Rightarrow \int\frac{dx}{x-a}=\int\frac{dz}{z-c}+logc_2$ where $logc_2$ is an integrating constant

$\Rightarrow \log(x-a) = \log(z-c) + logc_2$

$\Rightarrow \log(x-a) - \log(z-c) = \ logc_2$

$\Rightarrow logc_2 = log\frac{x-a}{z-c}$

$$\therefore c_2 = \frac{x-a}{z-c}$$

$\therefore The\ required\ general\ solution\ will\ be$

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi\left(\frac{x-a}{y-b}, \frac{x-a}{z-c}\right) = 0,\ \varphi\ is\ an\ arbitrary\ function.$ **Answer**

2. $\boldsymbol{xp + yq = z}$

Solution : Given PDE is,

$xp + yq = z$ …………………. (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ ……………………. (ii)

From 1st and 2nd fractions, we get

$$\frac{dx}{x} = \frac{dy}{y}$$

$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y} + \ logc_1$ where $logc_1$is an integrating constant.

$\Rightarrow logx = logy + logc_1$

$\Rightarrow logx - logy = \ logc_1$

$\Rightarrow logc_1 = log\frac{x}{y}$

$\Rightarrow c_1 = \frac{x}{y}$ …………… (iii)

From 1st and 3rd fractions, we get

$$\frac{dx}{x} = \frac{dz}{z}$$

$\Rightarrow \int \frac{dx}{x} = \int \frac{dz}{z} + \ logc_2$ where $logc_2$is an integrating constant.

$\Rightarrow logx = logz + logc_2$

$\Rightarrow logx - logz = \ logc_2$

$\Rightarrow logc_2 = log\frac{x}{z}$

$\Rightarrow c_2 = \frac{x}{z}$ .... (iv)

$\therefore The\ required\ general\ solution\ will\ be$

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi\left(\frac{x}{y}, \frac{x}{z}\right) = 0,\ \varphi\ is\ an\ arbitrary\ function.$ **Answer**

3. $\boldsymbol{p + q = 1}$

Solution : Given PDE is,

$p + q = 1$ .... (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$ .... (ii)

From 1st and 2nd fractions, we get

$dx = dy$

$\Rightarrow \int dx = \int dy + \ c_1$ where $c_1$is an integrating constant.

$\Rightarrow x = y + c_1$

$\Rightarrow c_1 = x - y$

From 2nd and 3rd fractions, we get

$dy = dz$

$\Rightarrow \int dy = \int dz + c_2$ where c_2 is an integrating constant.

$\Rightarrow y = z + c_2$

$\Rightarrow c_2 = y - z$

$\therefore The\ required\ general\ solution\ will\ be$

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi(x - y, y - z) = 0,\ \varphi\ is\ an\ arbitrary\ function.$ **Answer**

4. $x^2p + y^2q = z^2$

Solution : Given PDE is,

$x^2p + y^2q = z^2$ …………………. (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$ ……………………. (ii)

From 1st and 2nd fractions, we get

$\frac{dx}{x^2} = \frac{dy}{y^2}$

$\Rightarrow \int \frac{dx}{x^2} = \int \frac{dy}{y^2} + c_1$ where c_1 is an integrating constant.

$\Rightarrow -\frac{1}{x} = -\frac{1}{y} + c_1$

$\Rightarrow c_1 = \frac{1}{y} - \frac{1}{x}$

From 2nd and 3rd fractions, we get

$\frac{dy}{y^2} = \frac{dz}{z^2}$

$\Rightarrow \int \frac{dy}{y^2} = \int \frac{dz}{z^2} + c_2$ where c_2 is an integrating constant.

$\Rightarrow -\frac{1}{y} = -\frac{1}{z} + c_2$

$\Rightarrow c_2 = \frac{1}{z} - \frac{1}{y}$

$\therefore$ *The required general solution will be*

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0,\ \varphi \text{ is an arbitrary function.}$ **Answer**

5. $\boldsymbol{x^2p + y^2q + z^2 = 0}$

Solution : Given PDE is,

$x^2p + y^2q + z^2 = 0$

$\Rightarrow x^2p + y^2q = -z^2$………………… (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$ ……………………. (ii)

From 1st and 2nd fractions, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$\Rightarrow \int \frac{dx}{x^2} + c_1 = \int \frac{dy}{y^2}$ where c_1 is an integrating constant.

$\Rightarrow -\frac{1}{x} + c_1 = -\frac{1}{y}$

$\Rightarrow c_1 = \frac{1}{x} - \frac{1}{y}$

From 2nd and 3rd fractions, we get

$\frac{dy}{y^2} = \frac{dz}{-z^2}$

$\Rightarrow \int \frac{dy}{y^2} + c_2 = -\int \frac{dz}{z^2}$ where c_2 is an integrating constant.

$\Rightarrow -\frac{1}{y} + c_2 = \frac{1}{z}$

$\Rightarrow c_2 = \frac{1}{y} + \frac{1}{z}$

$\therefore$ *The required general solution will be*

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0,\ \varphi$ *is an arbitrary function.* **Answer**

6. $\boldsymbol{\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = sinx}$

Solution : Given PDE is,

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = sinx$$

$\Rightarrow p + q = sinx$ …………….. (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{sinx}$ ……………………. (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{1} = \frac{dy}{1}$$

$\Rightarrow dx = dy$

$\Rightarrow \int dx + c_1 = \int dy$ where c_1 is an integrating constant.

$\Rightarrow x + c_1 = y$

$\Rightarrow c_1 = y - x$ …………… (iii)

From 1st and 3rd fractions of (ii), we get

$$\frac{dx}{1} = \frac{dz}{sinx}$$

$\Rightarrow sinx\ dx = dz$

$\Rightarrow \int sinx\ dx + c_2 = \int dz$

$\Rightarrow -\cos x + c_2 = z$

$\Rightarrow c_2 = z + \cos x$ ……………… (iv)

$\therefore The\ required\ general\ solution\ will\ be$

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi(y - x, z + \cos x) = 0,\ \varphi\ is\ an\ arbitrary\ function.$ **Answer**

7. $\boldsymbol{yzp + 2xq = xy}$

Solution : Given PDE is,

$yzp + 2xq = xy$ ……………………… (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{yz} = \frac{dy}{2x} = \frac{dz}{xy}$ ……………………… (ii)

From 1st and 3rd fractions of (ii), we get

$$\frac{dx}{yz} = \frac{dz}{xy}$$

$\Rightarrow xdx = zdz$

$\Rightarrow \int xdx + \frac{c_1}{2} = \int zdz,$ where c_1 is an integrating constant.

$\Rightarrow \frac{x^2}{2} + \frac{c_1}{2} = \frac{z^2}{2}$

$\Rightarrow c_1 = z^2 - x^2$ ……….. (iii)

From 2nd and 3rd fractions of (ii), we get

$$\frac{dy}{2x} = \frac{dz}{xy}$$

$\Rightarrow ydy = 2dz$

$\Rightarrow \int ydy + \frac{c_2}{2} = 2\int dz,$ where c_2 is an integrating constant.

$\Rightarrow \frac{y^2}{2} + \frac{c_2}{2} = 2z$

$\Rightarrow c_2 = 4z - y^2$ ……………. (iv)

$\therefore$ *The required general solution will be*

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi(z^2 - x^2, 4z - y^2) = 0,$ *φ is an arbitrary function.* **Answer**

9. $yzp + zxq = xy$

Solution : Given PDE is,

$yzp + zxq = xy$ ………….. (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ ……………………. (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{yz} = \frac{dy}{zx}$$

$\Rightarrow xdx = ydy$

$\Rightarrow \int x dx + \frac{c_1}{2} = \int y dy,$ where $c_1$is an integrating constant.

$\Rightarrow \frac{x^2}{2} + \frac{c_1}{2} = \frac{y^2}{2}$

$\Rightarrow c_1 = y^2 - x^2$ ……….. (iii)

From 1st and 3rd fractions of (ii), we get

$$\frac{dx}{yz} = \frac{dz}{xy}$$

$\Rightarrow x dx = z dz$

$\Rightarrow \int x dx + \frac{c_2}{2} = \int z dz,$ where $c_2$is an integrating constant.

$\Rightarrow \frac{x^2}{2} + \frac{c_2}{2} = \frac{z^2}{2}$

$\Rightarrow c_2 = z^2 - x^2$ ……….. (iii)

$\therefore$ *The required general solution will be*

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi(y^2 - x^2, z^2 - x^2) = 0,$ *φ is an arbitrary function.* **Answer**

10. $\boldsymbol{zp = x}$

Solution : Given PDE is

$zp = x$ …………… (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{x}$ …………………….. (ii)

From 1st and 3rd fractions of (ii), we get

$$\frac{dx}{z} = \frac{dz}{x}$$

$\Rightarrow xdx = zdz$

$\Rightarrow \int xdx + \frac{c_1}{2} = \int zdz,$ where c_1 is an integrating constant.

$\Rightarrow \frac{x^2}{2} + \frac{c_1}{2} = \frac{z^2}{2}$

$\Rightarrow c_1 = z^2 - x^2$ ……….. (iii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{z} = \frac{dy}{0}$$

$\Rightarrow zdy = 0$

$\Rightarrow dy = 0$

$\Rightarrow \int dy = c_2,$ where c_2 is an integrating constant.

$\Rightarrow c_2 = y$ ……………….. (iv)

$\therefore$ *The required general solution will be*

$\varphi(c_1, c_2) = 0,$

$i.e., \varphi(z^2 - x^2, y) = 0,\ \varphi$ *is an arbitrary function.* **Answer**

11. $\boldsymbol{y^2p + x^2q = x^2y^2z^2}$

Solution : Given PDE is

$y^2p + x^2q = x^2y^2z^2$ …………… (i)

The Lagrange's auxiliary equations for (i) are

$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$ …………………….. (ii)

From 1st and 2nd fractions of (ii), we get

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$

$$\Rightarrow x^2dx = y^2dy$$

$\Rightarrow \int x^2dx + \frac{c_1}{3} = \int y^2dy,$ where $c_1$is an integrating constant.

$$\Rightarrow \frac{x^3}{3} + \frac{c_1}{3} = \frac{y^3}{3}$$

$\Rightarrow c_1 = y^3 - x^3$ ……….. (iii)

From 1st and 3rd fractions of (ii), we get

$$\frac{dx}{y^2} = \frac{dz}{x^2y^2z^2}$$

$$\Rightarrow x^2dx = \frac{dz}{z^2}$$

$\Rightarrow \int x^2dx + \frac{c_2}{3} = \int \frac{dz}{z^2},$ where $c_2$is an integrating constant.

$\Rightarrow \frac{x^3}{3} + \frac{c_2}{3} = -\frac{1}{z},$ where $c_2$is an integrating constant.

$$\Rightarrow x^3 + c_2 = -\frac{3}{z}$$

$\Rightarrow c_2 = -x^3 - \frac{3}{z}$……………….. (iv)

$\therefore \textit{The required general solution will be}$

$$\varphi(c_1, c_2) = 0,$$

$i.e., \varphi\left(y^3 - x^3, -x^3 - \frac{3}{z}\right) = 0,\ \varphi \textit{ is an arbitrary function.}$ **Answer**