Solutions of Linear Homogeneous Ordinary Differential Equations (when Right hand members is constant/function of x/exponential function) :

A few notations and their uses :

$$\frac{dy}{dx} = Dy$$

$$\frac{d^2y}{dx^2} = D^2y$$

$$\frac{d^3y}{dx^3} = D^3y$$

$$\left(\frac{1}{D}\right)x = D^{-1}x = \int x dx = \frac{x^2}{2}$$

$$\left(\frac{1}{D^2}\right)x = D^{-2}x = \frac{1}{D}\left\{\left(\frac{1}{D}\right)x\right\} = \frac{1}{D}\left(\frac{x^2}{2}\right)$$

$$= \frac{1}{2}\left\{\left(\frac{1}{D}\right)x^2\right\} = \frac{1}{2}.\frac{x^3}{3} = \frac{x^3}{6}$$

Particular Integral

Given equation is

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = X$$

$$\Rightarrow \ (D^2y + P_1Dy + P_2y) = X$$

$$\Rightarrow (D^2 + P_1D + P_2)y = X$$

$$\Rightarrow f(D)\text{y} = X$$

$$\Rightarrow y = \frac{1}{f(D)}X \qquad (\text{ P.I. })$$

The general solution is obtained as

y= C.F. + P.I.

Case (i) : Particular Integral when C.F. is the form x^m where m ∈ N

To find $\frac{1}{f(D)}x^m$, we write $[f(D)]^{-1}x^m$ and expanding $[f(D)]^{-1}$ in ascending powers of "D" by Binomial Theorems.

Some binomial expansions are given below :

(i) $(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 -$…………………..

(ii) $(1-D)^{-1} = 1 + D + D^2 + D^3 + D^4 +$…………………..

Example 1. Solve $(D^2+4)y = x^2$

Solution : Given, $(D^2+4)y = x^2 \ldots \ldots \ldots \ldots \ldots (i)$

Let us consider, $(D^2+4)y = 0$ ……………….. (ii)

Let us consider, $y = e^{mx}$ is a trial solution of (ii). Then, we get,

$(m^2+4)e^{mx} = 0$

$\Rightarrow m^2 + 4 = 0$ $[e^{mx} \neq 0]$

$\Rightarrow m^2 = -4$

$\Rightarrow m = 0 \pm 2i$

$\therefore \mathit{The\ complementary\ Function} = (A\cos 2x + B\sin 2x)$

(i) implies that

Particular Integral $= \frac{1}{D^2+4}x^2$

$= \frac{1}{4\left(1+\frac{D^2}{4}\right)}x^2$

$$= \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1} x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} + \frac{D^4}{16} - \cdots \ldots\right] x^2$$

$$= \frac{1}{4}x^2 - \frac{D^2}{16}x^2 + \cdots$$

$$= \frac{1}{4}x^2 - \frac{1}{16}.2$$

$$= \frac{1}{4}x^2 - \frac{1}{8}$$

$\therefore$ *The general solution*, $y = C.F. + P.I.$

$y = \ A\cos 2x + B\sin 2x + \frac{1}{4}x^2 - \frac{1}{8}$ ***Answer***.

Exercise XVIII(B)

[Recommended Text Book : Integral Calculus by Das & Mukherjee]

Solve the following equations :

1.(i) $\frac{d^2y}{dx^2} + 4y = 2x + 3$

Solution : Given equation is

$$\frac{d^2y}{dx^2} + 4y = 2x + 3$$

$\Rightarrow (D^2 + 4)y = 2x + 3$ ……………….. (i)

The auxiliary equation of (i) will be

$m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m = 0 \pm 2i$

$\therefore$ C.F. $= A\cos 2x + B\sin 2x$

and, P.I. $= \frac{1}{D^2+4}(2x+3)$

$$= \frac{1}{4\left(1+\frac{D^2}{4}\right)}(2x+3)$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}(2x+3)$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{D^4}{16}-\cdots\ldots\ldots\ldots\right](2x+3)$$

$$= \frac{1}{4}(2x+3)$$

$\therefore$ *The general solution,* $y = C.F. + P.I.$

$y = A\cos 2x + B\sin 2x + \frac{1}{4}(2x+3)$ ***Answer.***

(ii) $\frac{d^2y}{dx^2} + y = x^3$

Solution : Given equation is

$$\frac{d^2y}{dx^2} + y = x^3$$

$\Rightarrow (D^2+1)y = x^3$ ……………….. (i)

The auxiliary equation of (i) will be

$m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = 0 \pm i$

$\therefore$ C.F. $= A\cos x + B\sin x$

and, P.I. $= \frac{1}{D^2+1}x^3$

$$= (1 + D^2)^{-1} x^3$$

$$= [1 - D^2 + D^4 - \cdots \ldots \ldots \ldots] x^3$$

$$= x^3 - 6x$$

$\therefore$ *The general solution*, $y = C.F. + P.I.$

$y = A\cos x + B\sin x + x^3 - 6x$ ***Answer***.

2.(i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$

Solution : Given equation is

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$$

$\Rightarrow (D^2 + 2D)y = x^2$ ……………….. (i)

The auxiliary equation of (i) will be

$m^2 + 2m = 0$

$\Rightarrow m(m+2) = 0$

$\Rightarrow m = 0, -2$

$\therefore$ C.F. $= c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$

and, P.I. $= \frac{1}{(D^2+2D)} x^2$

$$= \frac{1}{D(D+2)} x^2$$

$$= \frac{1}{2D\left(1+\frac{D}{2}\right)} x^2$$

$$= \frac{1}{2} . \frac{1}{D} . \left(1 + \frac{D}{2}\right)^{-1} . x^2$$

$$= \frac{1}{2} . \frac{1}{D} . \left(1 - \frac{D}{2} + \frac{D^2}{4} - \cdots \ldots \ldots .\right) . x^2$$

$$= \frac{1}{2} . \frac{1}{D} . \left(x^2 - \frac{2x}{2} + \frac{2}{4} \right)$$

$$= \frac{1}{2} . \frac{1}{D} \left(x^2 - x + \frac{1}{2} \right)$$

$$= \frac{1}{2} . \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2} \right)$$

$$= \frac{1}{2} . \frac{2x^3 - 3x^2 + 3x}{6}$$

$$= \frac{2x^3 - 3x^2 + 3x}{12}$$

$\therefore$ *The general solution*, $y = C.F. + P.I.$

$$y = c_1 + c_2 e^{-2x} + \frac{2x^3 - 3x^2 + 3x}{12}$$ ***Answer***.

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = x$

Solution : Given equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = x$$

$\Rightarrow (D^2 + D - 6)y = x$ ………………. (i)

The auxiliary equation of (i) will be

$m^2 + m - 6 = 0$

$\Rightarrow (m + 3)(m - 2) = 0$

$\Rightarrow m = -3, 2$

$\therefore$ C.F. $= c_1 e^{-3x} + c_2 e^{2x}$

and, P.I. $= \frac{1}{(D^2 + D - 6)} x$

$$= \frac{1}{(D + 3)(D - 2)} x$$

$$= \frac{1}{5}\left[\frac{1}{D-2} - \frac{1}{D+3}\right] x$$

$$= \frac{1}{5}\left[\frac{1}{-2\left(1-\frac{D}{2}\right)} - \frac{1}{3\left(1+\frac{D}{3}\right)}\right] x$$

$$= \frac{1}{-10}\left(1 - \frac{D}{2}\right)^{-1} . x - \frac{1}{15}\left(1 + \frac{D}{3}\right)^{-1} . x$$

$$= \frac{-1}{10}\left(1 + \frac{D}{2} + \frac{D^2}{4} + \cdots\right) x - \frac{1}{15}\left(1 - \frac{D}{3} + \frac{D^2}{9} - \cdots\right) x$$

$$= \frac{-1}{10}\left(x + \frac{1}{2}\right) - \frac{1}{15}\left(x - \frac{1}{3}\right)$$

$$= \frac{-5x}{30} - \frac{5}{180}$$

$$= \frac{-x}{6} - \frac{1}{36}$$

$\therefore$ *The general solution*, $y = C.F. + P.I.$

$y = c_1 e^{-3x} + c_2 e^{2x} - \frac{x}{6} - \frac{1}{36}$ ***Answer***.

3.(i) $(D+3)^2 y = 25e^{2x}$

Solution : Given equation is

$(D+3)^2 y = 25e^{2x}$ ………………. (i)

The auxiliary equation of (i) will be

$(m+3)^2 = 0$

$\Rightarrow m = -3, -3$

$\therefore$ C.F. $= (c_1 + c_2 x)e^{-3x}$

and, P.I. $= \frac{1}{(D+3)^2} 25e^{2x}$

$$= \frac{25e^{2x}}{(2+3)^2}$$

$$= \frac{25e^{2x}}{25}$$

$$= e^{2x}$$

$\therefore$ *The general solution,* $y = C.F. + P.I.$

$y = (c_1 + c_2 x)e^{-3x} + e^{2x}$ ***Answer***

(ii) $(D^2 + 9)y = 9e^{3x}$

Solution : Given equation is

$(D^2 + 9)y = 9e^{3x}$ ……………… (i)

The auxiliary equation of (i) will be

$m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

$\therefore$ C.F. $= A\cos 3x + B\sin 3x$

and, P.I. $= \frac{1}{D^2+9} 9e^{3x}$

$$= \frac{9e^{3x}}{3^2+9}$$

$$= \frac{9e^{3x}}{18}$$

$$= \frac{e^{3x}}{2}$$

$$= \frac{1}{2} e^{3x}$$

$\therefore$ *The general solution,* $y = C.F. + P.I.$

$y = A\cos 3x + B\sin 3x + \frac{1}{2} e^{3x}$ ***Answer***.

4. (i) $\frac{d^2y}{dx^2} - a^2y = e^{ax}$

Solution : Given equation is

$$\frac{d^2y}{dx^2} - a^2y = e^{ax}$$

$\Rightarrow (D^2 - a^2)y = e^{ax}$ ……………… (i)

The auxiliary equation of (i) will be

$m^2 - a^2 = 0$

$\Rightarrow m = \pm a$

$\therefore$ C.F. $= c_1e^{ax} + c_2e^{-ax}$

and, P.I. $= \frac{1}{D^2-a^2} e^{ax}$

$= e^{ax}.\frac{1}{D^2}.1$

$= e^{ax}.\frac{x^2}{2}$

$= \frac{x^2e^{ax}}{2}$

$\therefore$ *The general solution,* $y = C.F. + P.I.$

$y = c_1e^{ax} + c_2e^{-ax} + \frac{x^2e^{ax}}{2}$ ***Answer.***

(ii) $\frac{d^2y}{dx^2} - y = e^{2x}$

Solution : Given equation is

$$\frac{d^2y}{dx^2} - y = e^{2x}$$

$\Rightarrow (D^2 - 1)y = e^{2x}$ ……………. (i)

The auxiliary equation of (i) will be

$m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x}$

$and,$ P.I. $= \frac{1}{D^2-1} e^{2x}$

$= \frac{e^{2x}}{2^2-1}$

$= \frac{e^{2x}}{3}$

$\therefore$ *The general solution,* $y = C.F. + P.I.$

$y = c_1 e^x + c_2 e^{-x} + \frac{e^{2x}}{3}$ ***Answer.***

(iii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^{3x}$

Solution : Given equation is

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^{3x}$$

$\Rightarrow (D^2 - 4D + 3)y = 2e^{3x}$ …………………….. (i)

The auxiliary equation of (i) will be

$m^2 - 4m + 3 = 0$

$\Rightarrow (m-3)(m-1) = 0$

$\Rightarrow m = 3,1$

$\therefore C.F. = c_1 e^{3x} + c_2 e^x$

And, $P.I. = \frac{1}{D^2-4D+3}.2e^{3x}$

$$= \frac{1}{(D-3)(D-1)}.2e^{3x}$$

$$= \frac{1}{2}\left(\frac{1}{D-3} - \frac{1}{D-1}\right)2e^{3x}$$

$$= \frac{1}{D-3}e^{3x} - \frac{1}{D-1}e^{3x}$$

$$= e^{3x}.\frac{1}{D}.1 - \frac{e^{3x}}{3-1}$$

$$= e^{3x}.x - \frac{e^{3x}}{2}$$

$$= xe^{3x} - \frac{e^{3x}}{2}$$

$\therefore$ *The general solution,* $y = C.F. + P.I.$

$y = c_1e^{3x} + c_2e^{x} + xe^{3x} - \frac{e^{3x}}{2}$ ***Answer.***