Interpolation Formulae

2.1 Introduction :

Suppose, $y = f(x)$ is a given function. Then, $f(x_0), f(x_1), f(x_2)$, ………., $f(x_n)$ are the corresponding values for the values of independent variable x, i.e., x_0, x_1, x_2, ………., x_n respectively with respect to the given function. When, we have to find out some intermediate terms or values of a given function with given set of values (both arguments and entries), we apply some special processes which are called Interpolations.

Interpolations are applied for both with Equal and Unequal Intervals.

2.2 Newton-Gregory Formula for Forward Interpolation with Equal Intervals :

Suppose, $y = f(x)$ is a given function.

Suppose, $f(a), f(a+h), f(a+2h)$, ………. , $f(a+nh)$ are $(n+1)$ corresponding values for $(n+1)$ equidistant values $a, a+h, a+2h$,………. , $a+nh$ of the independent variable x respectively, the interval of differencing is h.

Now, we consider, $f(x)$ is a nth degree polynomial in x, such that

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \ldots\ldots + A_n(x-a)(x-a-h)\ldots\ldots(x-a-\overline{n-1}\,h) \qquad \ldots\ldots\ldots\ldots (1)$$

where $A_0,\ A_1,\ A_2, \ldots\ldots\ldots, A_n$ are constants to be determined.

Putting successively $x = a, a+h, a+2h, \dots\dots\dots, a+nh$ in (1), we have,

$$f(a) = A_0 \quad (\,when\; x = a\,) \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

$$f(a+h) = A_0 + A_1 h \qquad (\,when\; x = a+h\,)$$

$\Rightarrow \quad f(a+h) = f(a) + A_1 h$

$\Rightarrow \quad A_1 h = f(a+h) - f(a)$

$\Rightarrow \quad A_1 = \frac{f(a+h)-f(a)}{h}$

$\Rightarrow \quad A_1 = \frac{\Delta f(a)}{h} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$

$$f(a+2h) = A_0 + A_1.2h + A_2.2h.h$$

$\Rightarrow \quad f(a+2h) = f(a) + \frac{\Delta f(a)}{h}.2h + A_2.2h^2$

$\Rightarrow \quad f(a+2h) = f(a) + 2[f(a+h) - f(a)] + A_2.2h^2$

$\Rightarrow \quad f(a+2h) = 2f(a+h) - f(a) + A_2.2h^2$

$\Rightarrow \quad A_2.2h^2 = f(a+2h) - 2f(a+h) + f(a)$

$\Rightarrow \quad A_2.2h^2 = \Delta^2 f(a)$

$\Rightarrow \quad A_2 = \frac{\Delta^2 f(a)}{2h^2} = \frac{\Delta^2 f(a)}{2\,!\,h^2} \qquad \dots\dots\dots\dots\dots\dots\dots (4)$

Proceeding in this way, we get

$$A_3 = \frac{\Delta^3 f(a)}{3\,!\,h^3}$$

$$A_4 = \frac{\Delta^4 f(a)}{4\,!\,h^4}$$

……………………

……………………

$$A_n = \frac{\Delta^n f(a)}{n\,!\,h^n}$$

Now, substituting these values of $A_0, A_1, A_2, \dots\dots\dots, A_n$ in (1), we get

$$f(x) = f(a) + (x-a)\frac{\Delta f(a)}{h} + (x-a)(x-a-h)\frac{\Delta^2 f(a)}{2\,!\,h^2}$$

$$+ (x-a)\ (a-h)(x-a-2h)\frac{\Delta^3 f(a)}{3\,!\,h^3} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ (x-a)(x-a-h)\ldots\ldots\ldots(x-a-\overline{n-1}\,h)\frac{\Delta^n f(a)}{n\,!\,h^n} \quad \ldots\ldots\ldots\ (5)$$

Then, we put

$$x = a + hu$$

$$\Rightarrow \quad u = \frac{x-a}{h}$$

Then, the equation (5) becomes

$$f(a+hu) = f(a) + uh.\frac{\Delta f(a)}{h} + uh(uh-h)\frac{\Delta^2 f(a)}{2\,!\,h^2}$$

$$+ uh(uh-h)(uh-2h)\ \frac{\Delta^3 f(a)}{3\,!\,h^3} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ uh(uh-h)(uh-2h)\ \ldots\ldots(uh-\overline{n-1}\,\mathrm{h})\ \frac{\Delta^n f(a)}{n\,!\,h^n}$$

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2\,!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a) +$$

$$\ldots\ldots\ldots\ldots + \frac{u(u-1)(u-2)\ldots\ldots(u-n+1)}{n\,!}\Delta^n f(a) \quad \ldots\ldots\ldots\ldots\ (6)$$

This formula is known as Newton-Gregory Formula for Forward Interpolation with equal interval.

2.3 Worked out Examples :

Example (1) : Given that sin 45°=0.7071, sin 50°=0.7660, sin 55°=0.8192, sin 60°=0.8660. Find the value of sin 52°.

Solution : Let us consider the function as

$y = f(x) = sinx°$

Here, $a = 45, h = 5$ and $x = a + hu = 52$

$a + hu = 52$

⇨ $45 + 5u = 52$

⇨ $5u = 52 - 45 = 7$

⇨ $u = 1.4$

The difference table is

Argument x	Entry $f(x)$	1st Difference $\Delta f(x)$	2nd Difference $\Delta^2 f(x)$	3rd Difference $\Delta^3 f(x)$
45	0.7071			
		0.0589		
50	0.7660		−0.0057	
		0.0532		−0.0007
55	0.8192		−0.0064	
		0.0468		
60	0.8660			

Now applying Newton-Gregory Forward Interpolation Formula (Upto third differences), we get

$$f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2\,!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)$$

$$\therefore \quad f(52) = 0.7071 + 1.4 \text{ x } 0.0589 + \frac{1.4(1.4-1)}{2\,!} \text{ x } (-0.0057)$$

$$+ \frac{1.4(1.4-1)(1.4-2)}{3\,!} \text{ x } (-0.0007)$$

$$\Rightarrow sin52° = 0.7071 + 0.08246 - 0.0016 + 0.00004$$

$$\Rightarrow sin52° = 0.7880 \text{ (approximately)}$$

Hence, $sin52° = 0.7880$ (approximately)

Example (2) : If l_x represents the number of persons living at age x In a life table, find as accurately as the data will permit l_x for values of $x = 35, 45\ and\ 47$. Given,

$l_{20} = 512, \quad l_{30} = 439, \quad l_{40} = 346, \quad l_{50} = 243.$

Solution : Given that

$$l_{20} = 512, \quad l_{30} = 439, \quad l_{40} = 346, \quad l_{50} = 243.$$

(i) First, we have to find out the value, when $x = 35$ i.e., l_{35}

Here, $a = 20, h = 10$ and $x = a + hu = 35$

$a + hu = 35$

⇨ $20 + 10u = 35$

⇨ $10u = 35 - 20 = 15$

⇨ $u = 1.5$

The difference table is

Argument x	Entry l_x	1st Difference Δl_x	2nd Difference $\Delta^2 l_x$	3rd Difference $\Delta^3 l_x$
20	512			
		-73		
30	439		-20	
		-93		10
40	346		-10	
		-103		
50	243			

Now applying Newton-Gregory Forward Interpolation Formula (Upto third differences), we get

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2\,!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)$$

$$\therefore \quad f(35) = l_{35} = 512 + 1.5 \text{ x } (-73) + \frac{1.5(1.5-1)}{2\,!} \text{ x } (-20)$$

$$+ \frac{1.5(1.5-1)(1.5-2)}{3\,!} \text{ x } 10$$

$\Rightarrow\ l_{35} = 512 - 109.5 - 7.5 - 0.625$

$\Rightarrow\ l_{35} = 394.4$

Hence, $l_{35} = 394$, approximately

(ii) Secondly, we have to find out the value, when $x = 42$ i.e., l_{42}

Here, $a = 20, h = 10$ and $x = a + hu = 42$

$a + hu = 42$

⇨ $20 + 10u = 42$

⇨ $10u = 42 - 20 = 22$

⇨ $u = 2.2$

The difference table is

Argument x	Entry l_x	1st Difference Δl_x	2nd Difference $\Delta^2 l_x$	3rd Difference $\Delta^3 l_x$
20	512			
		−73		
30	439		−20	
		−93		10
40	346		−10	
		−103		
50	243			

Now applying Newton-Gregory Forward Interpolation Formula (Upto third differences), we get

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2\,!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)$$

$$\therefore \quad f(42) = l_{42} = 512 + 2.2 \text{ x } (-73) + \frac{2.2(2.2-1)}{2\,!} \text{ x } (-20)$$

$$+ \frac{2.2(2.2-1)(2.2-2)}{3\,!} \text{ x } 10$$

$\Rightarrow\ l_{42} = 512 - 160.6 - 26.4 + 0.88 = 325.88$

Hence, $l_{42} = 326$, approximately

(iii) Finally, we have to find out the value, when $x = 47$ i.e., l_{47}

Here, $a = 20, h = 10$ and $x = a + hu = 47$

$a + hu = 47$

⇨ $20 + 10u = 47$

⇨ $10u = 47 - 20 = 27$

⇨ $u = 2.7$

The difference table is

Argument x	Entry l_x	1st Difference Δl_x	2nd Difference $\Delta^2 l_x$	3rd Difference $\Delta^3 l_x$
20	512			
		−73		
30	439		−20	
		−93		10
40	346		−10	
		−103		
50	243			

Now applying Newton-Gregory Forward Interpolation Formula (Upto third differences), we get

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2\,!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)$$

$$\therefore \quad f(47) = l_{47} = 512 + 2.7 \text{ x } (-73) + \frac{2.7(2.7-1)}{2\,!} \text{ x } (-20)$$

$$+ \frac{2.7(2.7-1)(2.7-2)}{3\,!} \text{ x } 10$$

$\Rightarrow\ l_{47} = 512 - 197.1 - 45.9 + 5.355$

$\Rightarrow\ l_{47} = 274.355$

Hence, $l_{47} = 274$, approximately

Example (3) : The values of $f(x)$ for $x = 0, 1, 2, \ldots\ldots, 6$ are given by

x	:	0	1	2	3	4	5	6
$f(x)$	:	5	7	10	15	25	49	99

Estimate the value of $f(2.2)$ using only five of the given values.

Solution : Here, last five values of $f(x)$ for $x = 2, 3, 4, 5, 6$ are taken into consideration so that 2.2 occurs in the beginning of the table.

Here, $a = 2, h = 1$ and $x = a + hu = 2.2$

$a + hu = 2.2$

⇨ $2 + u = 2.2$

⇨ $u = 2.2 - 2 = 0.2$

The difference table is

Argument x	Entry $f(x)$	1st Difference $\Delta f(x)$	2nd Difference $\Delta^2 f(x)$	3rd Difference $\Delta^3 f(x)$	4th Difference $\Delta^4 f(x)$
2	10				
		5			
3	15		5		
		10		9	
4	25		14		3
		24		12	
5	49		26		
		50			
6	99				

Now applying Newton-Gregory Forward Interpolation Formula (Upto fourth differences), we get

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2\,!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)$$
$$+\frac{u(u-1)(u-2)(u-3)}{4\,!}\Delta^4 f(a)$$

$$\therefore \quad f(2.2) = 10 + 0.2 \text{ x } 5 + \frac{0.2(0.2-1)}{2\,!} \text{ x } 5 + \frac{0.2(0.2-1)(0.2-2)}{3\,!} \text{ x } 9$$
$$+ \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4\,!} \text{ x } 3$$

$$\Rightarrow \quad f(2.2) = 10 + 1 - 0.4 + \frac{0.2(-0.8)(-1.8)}{6} \text{ x } 9$$
$$+ \frac{0.2(-0.8)(-1.8)(-2.8)}{24} \text{ x } 3$$

$$\Rightarrow \quad f(2.2) = 11 - 0.4 + 0.432 - 0.1008 = 10.9312$$

Hence, $f(2.2) = 10.93$ approximately

Example (4) : The following statistical data are given for 250 students who secured the marks in Mathematics in Higher Secondary Examination of a Junior College :

Marks Secured	Number of Students
30.....40	20
40.....50	25
50.....60	75
60.....70	100
70.....80	30

How many students secured more than 45 marks ?

Solution : Given data are considered as

Marks Secured	Number of Students
Less than 40	20
Less than 50	45
Less than 60	120
Less than 70	220
Less than 80	250

Here, $a = 40, h = 10$ and $x = a + hu = 55$

$a + hu = 55$

⇨ $40 + 10u = 55$

⇨ $10u = 55 - 40 = 15$

⇨ $u = 1.5$

The difference table is

Marks secured less than x	Number of students $f(x)$	1st Difference $\Delta f(x)$	2nd Difference $\Delta^2 f(x)$	3rd Difference $\Delta^3 f(x)$	4th Difference $\Delta^4 f(x)$
40	20				
		25			
50	45		50		
		75		-25	
60	120		25		-70
		100		-95	
70	220		-70		
		30			
80	250				

Now applying Newton-Gregory Forward Interpolation Formula (Upto fourth differences), we get

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2\,!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4\,!}\Delta^4 f(a)$$

$$\therefore \quad f(55) = 20 + 1.5 \text{ x } 25 + \frac{1.5(1.5-1)}{2\,!} \text{ x } 50 + \frac{1.5(1.5-1)(1.5-2)}{3\,!} \text{ x } (-25) + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4\,!} \text{ x } (-70)$$

$$\Rightarrow \quad f(55) = 20 + 37.5 + 18.75 + 1.5625 - 1.64062$$

$$\Rightarrow \quad f(55) = 76.17$$

Hence, Number of students who secured less than 55 is 76.