1.1 Algebra of Operator Δ :

(a) Difference of a constant is zero, i.e., $\Delta k = 0$, where k is a constant.
(b) The constant k and Δ are commutative, i.e., $\Delta k \equiv k\Delta$.
(c) The operator Δ is distributive over f and g, i.e.,

$$\Delta (f + g) \equiv \Delta f + \Delta g$$

(d) If k_1 and k_2 are constants, then

$$\Delta (k_1 f + k_2 g) = k_1 \Delta f + k_2 \Delta g$$

(e) nth difference of a nth degree polynomial is constant, $\Delta (ax^n + bx^{n-1} + \ldots\ldots\ldots\ldots + kx + l) = ah^n (n!)$, where $a \neq 0$ and h being the interval of differencing.

1.2 Worked out Examples :

Example (1) : Evaluate the following problems (h being the interval of differencing)

(i) $\Delta \, log x$
(ii) $\Delta \tan^{-1} x$
(iii) $\Delta \, sin 2x cos x$
(iv) $\Delta \, k^x$, where k is constant
(v) $\Delta \tan kx$, where k is constant
(vi) $\Delta \left[\frac{2^x}{x\,!}\right]$, h being unity.
(vii) $\Delta \sinh kx$, where k is constant
(viii) $\Delta \log \, g(x)$
(ix) $\Delta \, cot 3^x$
(x) $\Delta \, (ab^{kx})$, where a, b, k are constants.

Solution :

(i)
$$\Delta \, log x = log(x + h) - log x = log \frac{x+h}{x} = log\left(1 + \frac{h}{x}\right)$$

(ii)
$$\Delta \tan^{-1} x = \tan^{-1}(x + h) - \tan^{-1} x = \tan^{-1}\left[\frac{(x+h)-x}{1+(x+h)x}\right] = \tan^{-1} \frac{h}{1+(x+h)x}$$

$$\because \qquad \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$$

(iii) $\Delta\, sin2xcosx$

$$= \Delta[\tfrac{1}{2}(\sin 3x + \sin x)]$$

$$= \tfrac{1}{2}[\,\Delta \sin 3x + \Delta\, sin\ x\,]$$

$$= \tfrac{1}{2}[\{sin\, 3(x+h) - \ sin\ 3x\} + \{sin(x+h) - \ sin\ x\}]$$

$$= \tfrac{1}{2}[\,2\ cos\,\frac{3(x+h)+3x}{2}\, sin\,\frac{3(x+h)-3x}{2}$$

$$+ 2\ cos\,\frac{(x+h)+x}{2}\, sin\,\frac{(x+h)-x}{2}\,]$$

$$= \cos\left(3x + \frac{3h}{2}\right) sin\,\frac{3h}{2} + \cos\left(x + \frac{h}{2}\right) sin\,\frac{h}{2}$$

(iv) $\Delta\, k^x$, where k is constant

$$= k^{(x+h)} - k^x$$

$$= k^x k^h - k^x$$

$$= k^x(k^h - 1)$$

$$= k^x(k-1), \quad \text{when } h = 1$$

(v) $\Delta \tan kx$, where k is constant

$$= \tan k(x+h) - \tan kx$$

$$= \frac{\sin\ k(x+h)}{\cos\ k(x+h)} - \frac{\sin\ kx}{\cos\ kx}$$

$$= \frac{\sin\ k(x+h)\cos\ kx - \cos\ k(x+h)\ \sin\ kx}{\cos\ k(x+h)\cos\ kx}$$

$$= \frac{\sin\ \{\,k(x+h) - kx\,\}}{\cos\ k(x+h)\cos\ kx}$$

$$= \frac{\sin\ kh}{\cos\ k(x+h)\cos\ kx}$$

(vi) $\Delta\,[\frac{2^x}{x\,!}]$, h being unity.

$$= \frac{2^{x+1}}{(x+1)\,!} - \frac{2^x}{x\,!}$$

$$= \frac{x\,!\,2^{x+1} - 2^x(x+1)\,!}{(x+1)\,!\,x\,!}$$

$$= \frac{x\,!\,2^x 2 - 2^x(x+1)\,x\,!}{(x+1)\,!\,x\,!}$$

$$= \frac{x\,!\,2^x\{2-(x+1)\}}{(x+1)\,!\,x\,!}$$

$$= \frac{2^x(1-x)}{(x+1)\,!\,x\,!}$$

(vii) $\Delta \sinh kx$, where k is constant

$$= \sinh\ k(x+h) - \sinh\ kx$$

$$= 2\cosh \frac{k(x+h)+kx}{2}\, sinh \frac{k(x+h)-kx}{2}$$

$$= 2\cosh \frac{2kx+kh}{2}\, sinh \frac{kh}{2}$$

$$= 2\cosh\left(kx + \frac{kh}{2}\right) sinh \frac{kh}{2}$$

(viii) $\Delta \log\ g(\ x)$

$$= logg(x+h) - logg(x)$$

$$= log \frac{g(x+h)}{g(x)}$$

(ix) $\Delta\ cot3^x$

$$= \cot 3^{(x+h)} - cot3^x$$

$$= \frac{\cos\ 3^{x+h}}{sin3^{x+h}} - \frac{cos\ 3^x}{sin3^x}$$

$$= \frac{\sin\ 3^x \cos 3^{x+h} - \cos 3^x \sin 3^{x+h}}{sin\ 3^{x+h} \sin\ 3^x}$$

$$= \frac{\sin\ (3^x - 3^{x+h})}{sin\ 3^{x+h} \sin\ 3^x}$$

$$= \frac{\sin\ \{3^x(1-3^h)\}}{sin\ 3^{x+h} \sin\ 3^x}$$

(x) $\Delta\ (ab^{kx})$, where a, b, k are constants.

$$= ab^{k(x+h)} - ab^{kx}$$

$$= ab^{kx+kh} - ab^{kx}$$

$$= ab^{kx}b^{kh} - ab^{kx}$$

$$= ab^{kx}(b^{kh} - 1)$$

Example (2) : Evaluate the following problems (h being the interval of differencing and h =1)

(i) $\Delta^2 x^4$

(ii) $\Delta^4(ae^x)$

(iii) $\Delta^3[(1-ax)(1-bx)(1-cx)]$

(iv) $\Delta^{18}[(1-ax^3)(1-bx^4)(1-cx^5)(1-dx^6)]$

(v) $\Delta^2(ab^{kx})$

Solutions :

(i)
$$\begin{aligned}
\Delta^2 x^4 &= \Delta\,[\,\Delta x^4\,] \\
&= \Delta\,[(x+1)^4 - x^4\,] \\
&= \Delta\,(x+1)^4 - \Delta x^4 \\
&= \{(x+2)^4 - (x+1)^4\} - \{(x+1)^4 - x^4\} \\
&= (x+2)^4 - 2(x+1)^4 + x^4 \\
&= 12x^2 + 24x + 14
\end{aligned}$$

(ii)
$$\begin{aligned}
\Delta^4 ae^x &= \Delta^3[\,\Delta ae^x] \\
&= \Delta^3[\,ae^{x+1} - ae^x] \\
&= \Delta^3[\,a(e-1)e^x] \\
&= a(e-1)\Delta^2[\Delta e^x] \\
&= a(e-1)\Delta^2[e^{x+1} - e^x] \\
&= a(e-1)\Delta^2[e^x(e-1)] \\
&= a(e-1)^2\Delta^2 e^x \\
&= a(e-1)^2\Delta\,[\Delta e^x] \\
&= a(e-1)^2\Delta\,[e^{x+1} - e^x] \\
&= a(e-1)^2\Delta\,[e^x(e-1)] \\
&= a(e-1)^3\Delta\, e^x \\
&= a(e-1)^3[e^{x+1} - e^x] \\
&= a(e-1)^3 e^x(e-1) \\
&= a(e-1)^4 e^x
\end{aligned}$$

(iii) $\Delta^3[(1-ax)(1-bx)(1-cx)]$
$$\begin{aligned}
&= \Delta^3(-ax)(-bx)(-cx), \text{ other terms vanished} \\
&= \Delta^3(-abc)x^3 \\
&= (-abc).\,3\,! \\
&= -6abc
\end{aligned}$$

$$\Delta^n x^m = 0, \qquad if\ n > m$$
$$\Delta^n x^m = m\,!, \qquad if\ h = 1, n = m$$

(iv) $\Delta^{18}[(1-ax^3)(1-bx^4)(1-cx^5)(1-dx^6)$

$$= \Delta^{18}(-ax^3)(-bx^4)(-cx^5)(-dx^6)$$
$$=` \Delta^{18}(abcd)x^{18}$$
$$= abcd\ (18\,!\,)$$

(v) $\Delta^2(ab^{kx})$

$$= a\Delta^2 b^{kx}$$
$$= a\,[\,\Delta\,(\Delta b^{kx})]$$
$$= a\,[\,\Delta\,\left(b^{k(x+1)} - b^{kx}\right)]$$
$$= a\,[\,\Delta\,(b^k - 1)b^{kx}]$$
$$= a\,(b^k - 1)\Delta b^{kx}$$
$$= a\,(b^k - 1)\left(b^{k(x+1)} - b^{kx}\right)$$
$$= a\,(b^k - 1)(b^k - 1)b^{kx}$$
$$= a(b^k - 1)^2 b^{kx}$$

1.3 The Displacement Operator E :

The Operator E is known as Displacement or Shift Operator and it is defined as $Ef(x) = f(x+h)$ or $Ey_x = y_{x+h}$, where h is the interval of differencing.

In general,

$$E^2 f(x) = E[Ef(x)] = Ef(x+h) = f(x+2h)$$
$$E^3 f(x) = E[E^2 f(x)] = Ef(x+2h) = f(x+3h)$$

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Proceeding in this way, finally we get

$$\boldsymbol{E^n f(x) = f(x + nh)}$$

1.4 General Properties of the Operator E :

(i) The operator E is distributive over $f + g$ i.e.,

$$E(f+g) \equiv Ef + Eg$$

(ii) The constant k and the operator E are commutative, i.e.,

$$Ek \equiv kE$$

(iii) $E(k_1 f + k_2 g) \equiv k_1 Ef + k_2 Eg$, where $k_1$andk_2 are two constants.

(iv) The operator E follows the laws of indices

$$E^m E^n \equiv E^{m+n}$$

(v) The operators E and Δ are commutative

$$E\Delta \equiv \Delta E$$

1.5 Relations between the Operators Δ and E :

(i) $E \equiv 1 + \Delta$ or $\Delta \equiv E - 1$

(ii) $E^n \equiv (1 + \Delta)^n$

1.6 Relations between the Operators Δ, ∇ and E :

(i) $\nabla \equiv \Delta E^{-1}$

(ii) $\nabla \equiv 1 - E^{-1}$

1.7 Worked out Examples :

Example (1) : Evaluate the following problems, h being the interval of differencing :

(i) $\left(\frac{\Delta}{E}\right) x$, (ii) $\left(\frac{\Delta^2}{E}\right) x^2$, (iii) $\frac{E^2 x^2}{E x^2}$

Solutions :

(i)

$$\begin{aligned}\left(\frac{\Delta}{E}\right) x &= \frac{E-1}{E} x \\ &= \left(1 - \frac{1}{E}\right) x \\ &= x - E^{-1} x \\ &= x - (x - h) \\ &= h\end{aligned}$$

(ii)

$$\begin{aligned}\left(\frac{\Delta^2}{E}\right) x^2 &= \frac{(E-1)^2}{E} x^2 \\ &= \left(\frac{E^2 - 2E + 1}{E}\right) x^2\end{aligned}$$

$$= \quad (E - 2 + E^{-1})x^2$$
$$= \quad Ex^2 - 2x^2 + E^{-1}x^2$$
$$= \quad (x+h)^2 - 2x^2 + (x-h)^2$$
$$= \quad 2h$$

(iii) $$\frac{E^2x^2}{Ex^2} = \frac{(E-1)^2x^2}{Ex^2}$$
$$= \frac{(E^2-2E+1)x^2}{Ex^2}$$
$$= \frac{E^2x^2-2Ex^2+x^2}{Ex^2}$$
$$= \frac{(x+2h)^2-2(x+h)^2+x^2}{(x+h)^2}$$
$$= \frac{2h^2}{(x+h)^2}$$

Example (2) : Evaluate the following problems, h being the interval of differencing and $h = 1$

(i) $(\Delta + 1)(E + 1)(x + 1)$
(ii) $(\Delta - 1)(2\Delta + 3)(x^2 - 1)$
(iii) ΔE^2x^2
(iv) $(E^{-1}\Delta)x$

Solutions :

(i) $(\Delta + 1)(E + 1)(x + 1)$
$$= \quad \{(E-1)+1\}(E+1)(x+1)$$
$$= \quad E(E+1)(x+1)$$
$$= \quad (E^2+E)(x+1)$$
$$= \quad E^2(x+1) + E(x+1)$$
$$= \quad x + 3 + x + 2$$
$$= \quad 2x + 5$$

(ii) $(\Delta - 1)(2\Delta + 3)(x^2 - 1)$
$$= \quad \{(E-1)-1\}\{2(E-1)+3\}(x^2-1)$$
$$= \quad (E-2)(2E+1)(x^2-1)$$
$$= \quad (2E^2-3E-2)(x^2-1)$$
$$= \quad 2E^2(x^2-1) - 3E(x^2-1) - 2(x^2-1)$$

$$
\begin{aligned}
&= 2\{(x+2)^2-1\}-3\{(x+1)^2-1\}-2x^2+2 \\
&= -3x^2+2x+8
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\Delta E^2 x^2 &= \Delta\,(1+\Delta)^2 x^2 \\
&= \Delta\,(1+2\Delta+\Delta^2)x^2 \\
&= \Delta\, x^2 + 2\Delta^2 x^2 + \Delta^3 x^2 \\
&= (x+1)^2 - x^2 + 2.2\,! + 0 \\
&= 2x+1+4
\end{aligned}
$$

$$\Delta^2 x^2 = 2\,! \quad , \quad \Delta^3 x^2 = 0$$

(iv)

$$
\begin{aligned}
(E^{-1}\Delta)x &= E^{-1}(E-1)x \\
&= (1-E^{-1})x \\
&= x-(x-1) \\
&= 1
\end{aligned}
$$